Perpetuities and Annuities

(Welch, Chapter 03)

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Did you bring your calculator? Did you read these notes and the chapter ahead of time?

Maintained Assumptions

In this chapter, we maintain the assumptions of the previous chapter:

- We assume **perfect markets**, so we assume four market features:
  1. No differences in opinion.
  2. No taxes.
  3. No transaction costs.
  4. No big sellers/buyers—infinitely many clones that can buy or sell.
- We again assume **perfect certainty**, so we know what the rates of return on every project are.
- We again assume equal rates of returns in each period (year).

General Questions

- Are there any shortcut NPV formulas for long-term projects—at least under certain common assumptions?
- Or, do we always have to compute long summations for projects with many, many periods?
- Why do some of the folks in the room have the ability to quickly tell you numbers that would take you hours to figure out?
- How are loan payments (e.g., for mortgages) computed?

Specific Sample Questions

- If your firm produces \$5 million/year forever, and the interest rate is a constant 5% forever, what is the value of your firm?
- If your firm produces \$5 million/year in *real* (inflation-adjusted) terms forever, and the interest rate is a constant 5% forever, what is the value of your firm?
- What is the value of a firm that generates \$1 million in earnings per year and grows by the inflation rate?
- What is the monthly payment on a 6% 30-year fixed rate mortgage?

- NPV and Excel are a pain. Can't you teach us any shortcuts so that we can do the calculations in our heads as fast as the "quants" in our meeting?
  - You can think of perpetuities and annuities as shortcut formulas that can make computations a lot faster, and whose relative simplicity can sometimes aid intuition.

Simple Perpetuities

A perpetuity is a financial instrument that pays C dollars per period, forever. If the interest rate is constant and the first payment from the perpetuity arrives in period 1, then the PV of the perpetuity is:

$$(\mathsf{PV} =) \qquad \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

- Summation notation is *very common* in finance. It makes it easier if you are comfortable with its meaning! It is just notation, not really a new concept.
  More explanation: t is not an input variable; only C and r are. t is part of the notation that counts through terms. It's ephemeral.

Make sure you know when the first cash flow begins:

Tomorrow $[t = 1]$, not today $[t = 0]$!

I sometimes write CF_{+1}/r to remind myself of timing, even though cash flows are the same at time 1 as they are at time 25—I could have written CF_{25} instead.

Write out the perpetuity formula, $\sum_{t=1}^{\infty} \frac{C}{(1+r)^t}$

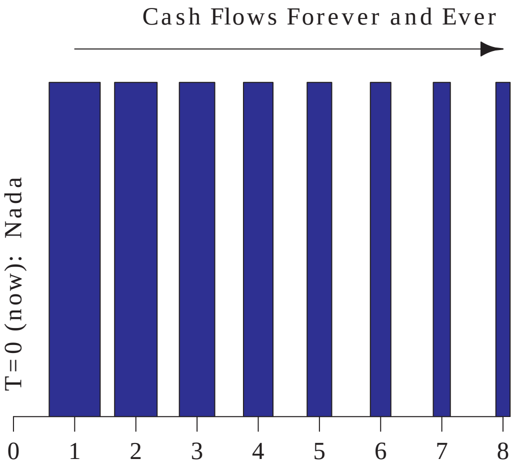
...in another language

- If you know how to program, the summation $\sum_{i=1}^{100} f(i)$ is the same as
  -
```
sum ⇐ 0.0
for i from 1 to 100 do
   begin
     sum ⇐ sum + f(i)
   end
return sum
```

    Again, note that **i** is not an input variable—instead, it is a device to indicate that we have 100 terms which we want to sum up. That is, after you have written out the formula, there is no **i** in it!
  - IMHO, programming teaches logical thinking. Basic computer programming is also useful in many jobs. Take at least an introductory programming course!

How can an infinite sum be worth less than $\$\infty$?

Because each future cash flow is worth a lot less than the preceding cash flow. In the graph, the PV of each cash flow is the bar's area.



What is the value of a promise to receive \$10 forever, beginning next year, if the interest rate is 5% per year?

What is the value of a promise to receive \$10 forever, beginning this year, if the interest rate is 5% per year?

What is the perpetuity formula if the first cash flow starts today rather than tomorrow?

Omitted Nerd Note: Time Consistency

Is the formula time-consistent? For example, if my house/property is paying up \$100 eternally, and I get cash, how can it still be worth the same tomorrow as it is today?

- The question is: if you have a perpetuity worth \$1,000, you will still have an annuity worth \$1,000 next year *and* get one payment, too. How can this be?

Next year, you will still have a perpetuity (then beginning the year thereafter, i.e., year 2). How much will the perpetuity be worth next year?

- Presume a cash flow of \$10 each year.
- Presume the interest rate is 10%.
- The perpetuity is thus worth \$100.
- Now, consider standing tomorrow.
  - You will still own a perpetuity.
  - It will then be worth \$1,000—but this is tomorrow.
  - So, today's value of tomorrow's perpetuity is $\$1,000/(1 + 10\%) \approx \$909$.

  In addition, you will get one extra cash flow of \$100 tomorrow, which is worth \$91.

Growing Perpetuities

A growing perpetuity pays CF, then $CF \cdot (1+g)$, then $CF \cdot (1+g)^2$, then ... For example, if $CF = \$100$ and $g = 0.10 = 10\%$, then you will receive the following payments:

$$
\begin{array}{rcll}
CF_0 = 0 & = & \$0 & \text{(no discount)} \\
CF_1 = \$100 & = & \$100.00 & \text{(then discount with } r_{0,1}\text{)} \\
CF_2 = \$100 \cdot (1+10\%) & = & \$110.00 & \text{(then discount with } r_{0,2}\text{)} \\
CF_3 = \$100 \cdot (1+10\%)^2 & = & \$121.00 & \text{(then discount with } r_{0,3}\text{)} \\
CF_4 = \$100 \cdot (1+10\%)^3 & = & \$133.10 & \text{(then discount with } r_{0,4}\text{)} \\
CF_5 = \$100 \cdot (1+10\%)^4 & = & \$146.41 & \text{(then discount with } r_{0,5}\text{)} \\
\text{and so on, forever} & & &
\end{array}
$$

The PV of a growing perpetuity can be quickly computed as

$$PV = \sum_{t=1}^{\infty} \frac{CF_1 \cdot (1+g)^{t-1}}{(1+r)^t} = \boxed{\frac{CF_1}{r-g}}$$

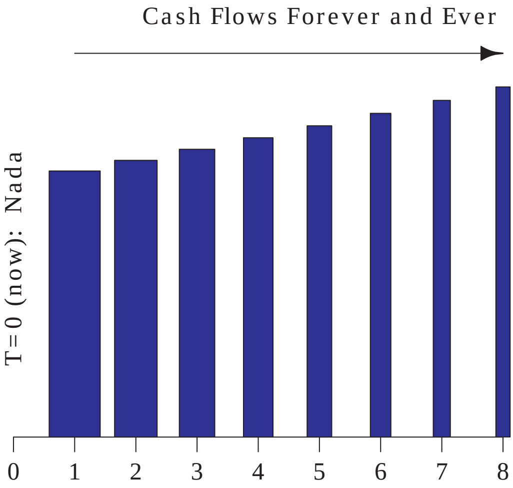
You must memorize the RHS formula, and know what it means!

- The growth term acts like a reduction in the interest rate.
- The time subscript for the payment matters now, because $C_1 \neq C_2 \neq C_t$.

Check the growing perpetuity formula by hand.

How can a **growing** infinite sum be less than infinite?

Because the growth is not fast enough. This is only the case if $r > g$.
The formula makes no sense if $r < g$.



What is the value of a promise to receive \$10 next year, growing by 2% (just the inflation rate) forever, if the interest rate is 6% per year?

What is the value of a firm that just paid \$10 **this** year, growing by 2% forever, if the interest rate is 5% per year?

What is the formula for the value of a firm which will only grow at the inflation rate, and which will have $1 million of earnings next year?

In 10 years, a firm will have annual cash flows of $100 million. Thereafter, its cash flows will grow at the inflation rate of 3%. If the applicable interest rate is 8%, estimate its value if you will sell the firm in 10 years? What would this be worth today?

Common Usage of Growing Perpetuity Formula

Growing perpetuity shortcuts are commonly used, and in many contexts. The most prominent use occurs in "pro-formas," where growing perpetuities are typically used to guestimate the present value of the residual firm value after an arbitrary T years in the future. A common long-run growth rate in this formula is then often the inflation rate. (The first T years are computed in more detail.) Typical T's in pro-formas are about 10 years.

What should be the share price of a firm that pays dividends of \$1/year, whose dividends have grown by 4% every year and will continue to do so forever, if its cost of capital is 12% per annum?

What is the cost of capital for a firm that pays a dividend yield of 5% per annum today, if its dividends are expected to grow at a rate of 3% per annum forever?

The Gordon Dividend Growth Model

- Using D for CF gives you the GDGM.
- **Don't trust the GDGM: Dividends can be rearranged.**
  - In fact, there is a fairly strong irrelevance proposition here. Given its underlying projects, it should not matter whether the firm pays out \$1 or \$10 in dividends. What it does not pay out in dividends today will make more hey (dividends) next year. Thus, expected rates of returns obtained from the Gordon model are highly suspect.
  - An improvement on the simple GDGM is to work out the plowback ratio, which takes into account that reinvested cash should pay more dividends in the future.
  - A better version, although without a fancy name, uses earnings instead of dividends. (There can be a similar irrelevance proposition for earnings as there is for dividends [firms can move earnings across periods], but it is not as easy/legal to shift earnings.)
- The GDGM is sometimes used to obtain an implied cost of capital, just as we did on the previous slide. Phrased differently, $r = g + D/P$ is the expected rate of return embedded in the price of the firm today. A higher price today implies a lower implied cost of capital at which the firms can obtain capital from investors.

In 2000, the P/E ratio of the stock market reached about $\boxed{45}$. If you assume that these corporations will grow roughly at the overall economy's (GDP) growth rate of 4–5% per year, what should investors have reasonably expected in terms of a likely future rate of return implied by the stock market's level?

Annuities

An annuity is a financial instrument that pays CF dollars for T years. It has the following PV formula:

$$PV = \sum_{t=1}^{T} \frac{CF}{(1+r)^t} = \left(\frac{CF}{r}\right) \cdot \left[1 - \frac{1}{(1+r)^T}\right]$$

- Again, make sure you know when the first cash flow begins: It starts tomorrow $[t = 1]$, not today $[t = 0]$!
- You should remember this formula, or at least be able to quickly derive it. I remember the formula, because an annuity is a perpetuity today, minus a (properly discounted) perpetuity in the future.

$$PV = \left(\frac{CF}{r}\right) - \frac{1}{(1+r)^T} \times \left(\frac{CF}{r}\right)$$

- Sometimes, if there are fewer than 10 terms, I am too lazy to use it. I just use a computer.

What is the value of a 3-decade annuity \$100 million each decade, with a decadal (10-year) interest rate of 50%, via the plain NPV formula and via the annuities formula. The first payment occurs in 10 years.

Annuity Example: Mortgage Loan

Here is a summary of how mortgage payments are usually calculated:

A 30-year mortgage is an annuity with 360 monthly payments, starting one month from today. Because payments are monthly, we need the monthly interest rate. The monthly rate on a mortgage is always computed as the quoted rate divided by 12. (In other words, like bank interest, your actual annual interest rate on a mortgage is higher than quoted. Lovely, isn't it?)

So the monthly interest rate on a 9% mortgage is

$$\mathsf{r}_{\mathsf{monthly}} = 0.09/12 = 0.0075 \text{ per month}$$

To buy a house, you need to take out a \$1,200,000 fixed rate mortgage with 30 years to maturity, 360 equal monthly payments, and a quoted interest rate of 9%. (You could also call this 9%, compounded monthly.) What will be your monthly mortgage payment?

Omitted: Principal and Interest Decompositions

Of the first month's payment, how much is interest and how much is principal? What is the balance remaining on the loan after 3 months? After 10 years? Uncle Sam and early repayment means you need to know how to do this calculation! In addition, you may be curious where the principal+interest numbers on your annual mortgage or student loan statements come from.

Month 1 The monthly interest rate is 0.75%, so the amount of interest due at the end of the first month is

$$0.0075 \cdot \$1,200,000 = \$9,000.00$$

Because \$9,000 of the first payment of $\approx \$9,655.47$ goes to paying interest, the remaining $\$9,655.47 - \$9,000 \approx \$655.47$ goes to paying off some of the remaining principal on the loan, so the balance on the loan at the end of one month, after making the first payment, is

$$\$1,200,000 - \$655.55 \approx \$1,199,344.53$$

Month 2 Interest charged during month 2 is

$$\$1,199,344.53 \cdot 0.0075 \approx \$8,995.08$$

So \$8,995.08 of month 2's payments goes to paying interest, and the remaining

$$\$9,655.47 - \$8,995.08 \approx \$660.39$$

goes to paying off principal, so the balance remaining on the loan after the month 2 payment is

$$\$1,199,344.53 - \$660.387 \approx \$1,198,684.14$$

Month 3

$$\text{Interest} \approx \$1,198,684.14 \cdot 0.0075 \approx \$8,990.13$$

$$\text{Principal Repayment} \approx \$9,655.47 - \$8,990.13 \approx \$665.34$$

$$\text{Remaining balance} \approx \$1,198,684.14 - \$665.34 \approx \$1,198,018.80$$

Month 120 For year 10, we could continue like this for 120 periods.
A simpler method (clever shortcut) is to remember that the remaining balance always equals the present value of the remaining payments, calculated using the loan's interest rate to do all discounting.
(You can compute the remaining balance in any way whatsoever. You might as well do this the slow way in Excel.)
After 10 years (120 months) there are 360 - 120 = 240 payments remaining. So

$$\text{Remaining Balance}_{\textit{after } 120 \text{ months}} = \frac{9,655.47}{1.0075} + \frac{9,655.47}{(1.0075)^2} + \ldots + \frac{9,655.47}{(1.0075)^{240}} \approx \$1,073,156.93$$

Assume the interest rate is $r = 20\%$. You need to rent a building. The landlord gives you two choices, a two-year lease with an extra payment in the first year, or a three-year lease:

Time	0	1	2
Project A (or Rent A)	\$20	\$12	
Project B (or Rent B)	\$15	\$15	\$15

Which is better?

What is the PV (or cost) of project A?

What is the “equivalent annual rent” of project A?

What if you need to use the building for exactly 2 years (and cannot rent it out to someone else to finish a 3-year lease)?

What if you need to use the building for exactly 3 years (and cannot rent it out to someone else beyond)?

Why is this EAC stuff in the annuities chapter?

Because the \$16.36 is the constant cash flow over 2 years.

Of course, rental cost and rental (project) income are mutual flip sides.

Level-Coupon Bonds

- Most bonds are coupon bonds, i.e., they have interim payments.
- Most corporate bonds are level-coupon bonds—this means that the coupon level will remain the same over the life of the bond.
- Most common bond: **x% semi-annual level coupon bond**.
  - Take the principal (often $1,000 for corporate bonds), multiply it by x% to obtain the annual coupon payment, divide it by two, and this is the coupon that is paid every six months.
  - For example, a 8% semi-annual level coupon bond pays $40 every six months on $1,000 in principal. At maturity, it pays $1,040.
  - **The x% is not the interest rate implicit in the bond!** It is a standard way to tell you the payment flow of this bond.
- The book works out a full example of level coupon pricing. This is tedious but straightforward. Make sure you can do it, too. We just cover some preliminaries here.

What are the payments to a 5% semi-annual level coupon bond, \$100 million, due in 2.5 years?

A zero bond has no interim payments. How do you earn interest on a bond that gives you no interest payments?

Is the coupon rate of a bond equal to the interest rate?

What is the interest rate on an IBM 3.5% semi-annual coupon bond?

An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years and sells for \$806,070. What is the implied interest rate (here called an IRR—more soon) that this insurance company is offering you?

An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years, growing at an "inflation-compensator" rate of 3%, and sells for \$806,070. What is the interest rate?

The prevailing interest rate is 10%/year. If you put aside \$1,000,000 to cover 18 years of expenses, how much could you draw down each year?

The prevailing interest rate is 10%/year. If you want to draw \$100,000 each year to cover 18 years of expenses, how much would you have to set aside?

Assume that our firm has stopped growing in real terms, and the current interest rate is 6% per annum. The inflation rate is 2% per annum. This year, we earned \$100,000. What is the value of the firm? (Do it over-the-envelope, and exact [what is the first cash flow?])

In 2016, GOOG (Google, called Alphabet nowadays) had a P/E earnings ratio of 27. Its cost of capital was roughly 8%. If Alphabet lasts forever, what does the market believe its growth rate will be?

An example drawn from an actual automobile loan agreement: The advertisement claimed,

> *12 month car loans. Only 9%!*

For this 12-month \$10,000 loan, at 9%, you owe \$10,900. Thus, your monthly payments will come out to $\$10,900/12 \approx \$908.33$ per month. OK?

What should be the PV of the car loan?

If you took out a loan from the bank at a true interest rate of 9% (8.649% quoted as compounded monthly), how much would the bank have asked you to pay each month?

Whence the difference? What is the actual interest rate (IRR) on the car dealer's proposed deal loan? Why is the IRR so high?

The true interest rate must set the present value of the payments equal to the initial loan. The dealer proposes

$$\$10,000 = \frac{\$908.33}{1 + \mathsf{r}_{\mathsf{monthly}}} + \frac{\$908.33}{(1 + \mathsf{r}_{\mathsf{monthly}})^2} + \ldots + \frac{\$908.33}{(1 + \mathsf{r}_{\mathsf{monthly}})^{12}}$$

This comes to $\mathsf{r}_{\mathsf{monthly}} \approx 1.35\%$. This is not 9% but an annual rate of

$$\mathsf{r}_{\mathsf{annual}} = (1 + \mathsf{r}_{\mathsf{monthly}})^{12} - 1 \approx 1.0135^{12} - 1 \approx \boxed{17.5\%}.$$

Each month you pay off part of the principal, thereby borrowing less later in the year. Yet the interest rate calculated by the dealer assumes that you borrow all $10,000 for the whole year.

Should you use perpetuities? How long will firms last?

What fraction of a perpetuity's value comes from the first t years? I.e., how reasonable an approximation is it to use a perpetuity in a meeting for a quick answer?

$$1 - \frac{1}{(1+r)^t}$$

So, this fraction is larger if r and t are bigger. For $r = 5\%$,

	$T = 20$	$T = 30$	$T = \infty$
r= 5%	62%	77%	100%
r=10%	85%	94%	100%

So, for high interest-rate (risky) cash flows, predicting 10–20 years out is mostly the same. This tells you how much you are off if you shortcut.

Omitted, but in the Book or Web Appendix

- Proof of Formulas.
- The formula for a growing annuity.
  - A growing annuity pays $\mathsf{CF} \cdot (1+g)^{t-2}$ per year starting in period 1 for T periods. Its present value is given by

$$\mathsf{PV} = \sum_{t=1}^{T} \left[\frac{(1+g)^{t-2}}{(1+r)^{t}} \right] \cdot \mathsf{CF} = \left(\frac{\mathsf{CF}}{\mathsf{R}-g} \right) \cdot \left[1 - \frac{(1+g)^{T}}{(1+r)^{T}} \right]$$

    If you do not wish to memorize the growing annuity formula...look it up in the book. It is what I do. Now you have seen it, and you know where to look it up. Finito.
  - Example Usage: You have an annuity that pays $100 in period 1. The annuity grows at a rate of 6% per year, and pays off until and including period 10. The discount rate equals 8%.