Time-Varying Rates of Return, Bonds, Yield Curves

(Welch, Chapter 05)

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Did you bring your calculator? Did you read these notes and the chapter ahead of time?

Maintained Assumptions

In this chapter, we maintain the assumptions of the previous chapter:

- We assume **perfect markets**, so we assume four market features:
  1. No differences in opinion.
  2. No taxes.
  3. No transaction costs.
  4. No big sellers/buyers—infinitely many clones that can buy or sell.
- We again assume **perfect certainty**, so we know what the rates of return on every project are.

- **But we no longer assume equal rates of returns in each period (year)!**
- Oranges cost more in the winter than in the summer. So why can't project payoffs not have different prices (rates of return) if they will realize at different times?

Time-Varying Rates of Returns

All earlier formulas hold.

- The only difference is that $(1 + r_{0,t}) \neq (1 + r)^t$.
- The main complication is that we now need many subscripts—one for each period. For example

$$(1 + r_{0,3}) = (1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})$$

$$\text{NPV} = C_0 + \frac{C_1}{(1 + r_{0,1})} + \frac{C_2}{(1 + r_{0,2})} + \frac{C_3}{(1 + r_{0,3})}$$

$$= C_0 + \frac{C_1}{(1 + r_{0,1})} + \frac{C_2}{(1 + r_{0,1}) \cdot (1 + r_{1,2})} + \frac{C_3}{(1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})}$$

- If you like it more formal,

$$(1 + r_{t,t+i}) = (1 + r_{t,t+1}) \cdot (1 + r_{t+1,t+2}) \cdots (1 + r_{t+i-1,t+i})$$

$$= (1 + r_{t+1}) \cdot (1 + r_{t+2}) \cdots (1 + r_{t+i}) = \prod_{j=t+1}^{t+i} (1 + r_j)$$

$$\text{PV} = \sum_{t=1}^{\infty} \left(\frac{\text{CF}_t}{(1 + r_{0,t})} \right) = \sum_{t=1}^{\infty} \left[\frac{\text{CF}_t}{\prod_{j=1}^{t} (1 + r_j)} \right]$$

- Recall that r_j is an abbrev for $r_{j-1,j}$.

...in another language

Here is a computer program that executes this formula. It relies on two subroutines, cashflow(time) and discountrate(timestart, timeend).

$\textbf{discountfactor} \leftarrow 1.0;$
$\textbf{npv} \leftarrow 0.0;$
for **timei**$=0$ *to* infinity *do*
    begin
        $\textbf{discountfactor} \leftarrow \textbf{discountfactor}/(1 + \text{discountrate}(\textbf{timei} - 1, \textbf{timei}))$
        $\textbf{npv} \leftarrow \textbf{npv} + \text{cashflow}(\textbf{timei}) * \textbf{discountfactor};$
    end
return **npv**;

Is 1,573 miles in 28.6 hours fast or slow?

Your project will give you a rate of return of 100% (double your money) over 15 years. Is this a lot or a little?

How does this 15-year rate of capital accumulation compare to a rate of capital accumulation of 1% over 3 months?

If the 1-year interest in year 1 is 5%, and the 1-year interest rate in year 2 will be 3%, what is the annualized interest rate?

Is an annualized interest rate more like an average or more like a sum?

It is more like an average. Indeed, it is called the geometric average!

Important

Almost all interest rates are quoted in annualized terms.

Annualized interest rates are (often just a little) below average interest rates, because they take (away / into account) the interest on interest.

Inflation: Real and Nominal Rates

- A nominal cash flow is simply the nominal number of dollars you pay out or receive.
- A real cash flow is adjusted for inflation. A real dollar always has the same purchasing power.
- If the U.S. were to call everything that is a cent today a dollar henceforth, instant inflation would be 9,900%—and yet it would not matter as long as all contracts today are clear about the units (dollars) and their translations.

  If properly contracted for, inflation is *not* a market imperfection.

  Just because quoted prices are less in Euros than in Lira can be called deflation, but it does not in itself create a problem.
  (If you need it even clearer, realize that a Euro is not the same as a Lira. In the same way, a Euro next year is not the same as a Euro this year.)
- In sum, inflation per se is not a friction (or market imperfection)—if everything is contracted in real terms. However, in the real world, most contracts are in nominal terms, so as an investor you must worry about inflation, and sometimes it can have similar effects.

An Example With Cash Flows and Inflation

- You have \$100, which you invest for 1 year at 10%.
- Bread sells for \$2.00 today.
- Your \$100 can purchase 50 loaves today.
- *Bread Inflation* over the next year will be 4%.
  - How is inflation (the CPI) defined?
- The bank pays a nominal rate of return of 10% per year.

What is your real rate of return?

What is the formula that relates the nominal rate, the real rate, and the inflation rate?

More about the inflation adjustment formula

More generally:

$$(1+0.0577)\cdot(1+0.04)\approx(1+0.10)$$

$$(1+\text{real rate})\cdot(1+\text{inflation rate})=(1+\text{nominal rate}).$$

You must remember this formula!

- **Intuition:** Why is this a "one-plus" type formula? Sorry, my intuition is not that good. I convince myself with examples here.
- When all rates are very small, the approximation

  $$\text{real rate}+\text{inflation rate}\sim\text{nominal rate}$$

  can be acceptable, *depending on the circumstances*, but this approximation formula is *not* exactly correct.
-
  - One real dollar today equals one nominal dollar today. (Usually!)
  - An inflation-adjusted dollar is $\$1/(1+\pi)$. So, \$110 next year is $\$110/1.04\approx\$105.77$ today in inflation-adjusted dollars. \$100 nominal next year is \$96.15 real dollars today.
  - Sometimes, real dollars are also called "inflation-adjusted" dollars, or—and this is where it gets real awful—are even called "in today's dollars." Unfortunately, different people mean difference things by these phrases. *In case of doubt, ask!!*

If a project will return \$110 in nominal cash next year, and the cost of capital is 10%, what is the PV?

If the inflation rate is 4%, and a project will return \$110 in nominal cash next year, then what is the purchasing power of this future \$110 in today's *real* dollars?

If the inflation rate is 4%, and the cost of capital is 10%, then what is the *real* cost of capital?

What is the project's *real* dollar value discounted by the *real* cost of capital? Why?

Important

Either discount nominal dollars with nominal interest rates,
or discount real dollars with real interest rates.

Never mix nominal cash flows with real rates.

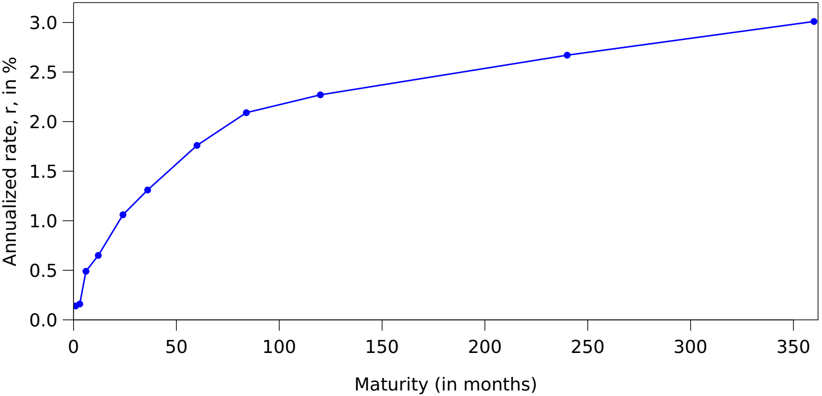
What is today's interest rate?

What is the inflation rate today?

The Yield Curve and US Treasuries

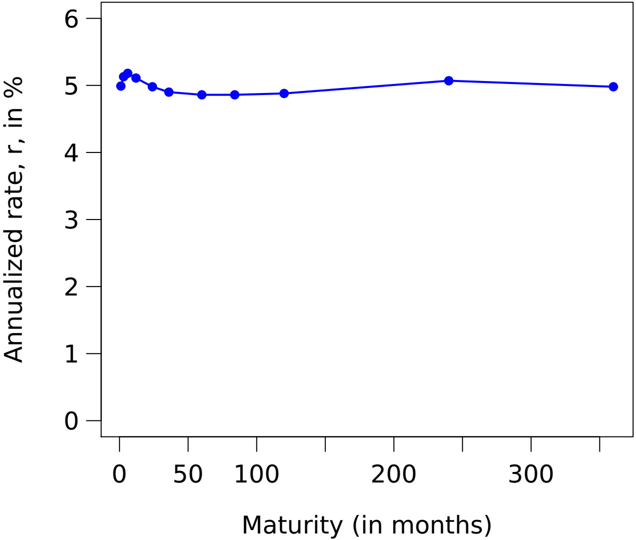
- US Treasuries are the most important financials security.
  - The outstanding amount was ≈\$18 trillion in 2015.
  - Annual trading is ≈\$100-\$150 trillion. (Turnover = 5-10 Times!)
  - Names: Bills (–0.99yr), Notes (1yr–10yr), Bonds (10yr–).
  - (Only the mortgage bond market is bigger than the UST market.)
- This market is close to "perfect":
  - Extremely low transaction costs (for traders).
  - Few opinion differences (inside information).
  - Deep market—many buyers and sellers.
  - Income taxes depend on owner.
- In addition, there is (almost) no uncertainty about repayment. (PS: a market could still be perfect, even if payoffs are uncertain.)
- (Zero-coupon) US Treasuries are among the simplest possible financial instrument in the world.
- **The yield curve is the plot of annualized yields (Y-axis) against time-to-maturity (X-axis).**

Yield Curve Dec 2015

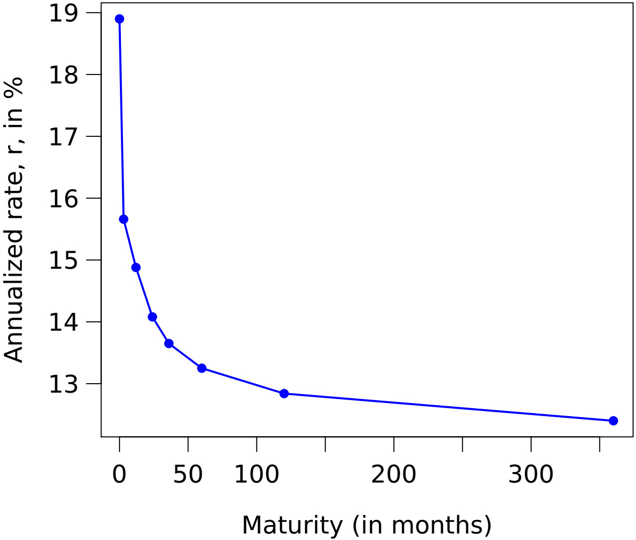


Can the Treasury yield curve be flat? Can it slope down?

Yield Curve Jan 2007



Yield Curve Dec 1980



Time-Varying Cost of Capital

The yield curve is a fundamental tool of finance. It always graphs annualized rates. It measures differences in the costs of capital for (risk-free) projects with different horizons.

In the real world, many variations on the yield curve are in use, e.g., yield-curves constructed from risky corporate bonds or yield-curves constructed from foreign bonds.

If not further qualified, when someone talks about the yield curve, they mean the yield curve on US Treasuries.

Is the 3-year bond a better deal than the 1-year bond?

What is the most common yield curve shape?

What does an upward sloping or downward sloping yield curve mean for the economy (not for an investor)?

What are today's short-term interest rates? How do they compare to the inflation rate? What does it mean for a taxed retail investor to hold short-term bonds?

Does the Fed control the (Treasury) yield curve?

Names: Spot and Forward Rates

- We call a currently prevailing interest rate for an investment starting today a *spot* interest rate.
- A forward rate is an interest rate that will begin with a cash flow in the future. It is the opposite of a spot rate.
- Like all other interest rates, spot and forward rates are usually quoted in annualized terms.

What is the annualized spot rate on a 1-month US T-bill today?

What is the annualized spot rate on a 30-year US T-bond today?

What does the yield curve today imply about future interest rates? Can you lock in future interest rates today?

More Painful Notation

- We denote an annualized interest rate over 15 years as $r_{\overline{15}}$. This contrasts with our notation for the 15-year non-annualized holding interest rate, which is $r_{0,15}$.

$$(1+r_{\overline{15}})^{15} \equiv (1+r_{0,15}) \equiv (1+r_{0,1}) \cdot (1+r_{1,2}) \cdot \ldots \cdot (1+r_{14,15})$$

$$\iff (1+r_{\overline{15}}) \equiv (1+r_{0,15})^{1/15}$$

$$(1+r_{\overline{t}})^{t} \equiv (1+r_{0,t})$$

$$\iff (1+r_{\overline{t}}) \equiv (1+r_{0,t})^{1/t}$$

$$\textbf{Example: } r_{0,5} = 27.63\% \iff r_{\overline{5}} = 5\% \; .$$

  - This is our notation, and not necessarily used elsewhere. To make matters worse, some people will use R to mean $1+r$, believing you can figure out whatever they may have meant. Others will just capitalize R and mean the same thing, namely r. Sigh...

- Notation Summary:

$$(1+r_{0,1}) = (1+r_{\overline{1}})^{1} = (1+r_{0,1})$$

$$(1+r_{0,2}) = (1+r_{\overline{2}})^{2} = (1+r_{0,1}) \cdot (1+r_{1,2})$$

$$(1+r_{0,3}) = (1+r_{\overline{3}})^{3} = (1+r_{0,1}) \cdot (1+r_{1,2}) \cdot (1+r_{2,3})$$

- The interest rate from period 1 to period 2 is called the *1-Year Forward (Interest) Rate from Period 1 to Period 2*.
- In a world of certainty, the forward rate will be the future spot rate: We know it! (Later I will show you how you can make the forward rate your personal future spot rate, even in a world of uncertainty.)

Approximate Answers

Remember:

- An annualized rate of return is more like an average.
- A holding rate of return is more like the sum.

A 1-year bond has an (annual) rate of return of 5%. When the first bond will come due, you will be able to purchase another 1-year bond that will have an (annual) rate of return of 10%. When the second bond will come due, you will be able to purchase another 1-year bond that will have an (annual) rate of return of 15%.

What are the three total holding rates of returns, and the three annualized rates of return?

(Calculator VERBOTEN. Use your intuition.)

Rates of Return		
Spot + Forward	Holding	Annualized
$r_{0,1} = 5\%$	$r_{0,1} =$	$r_{\overline{1}} =$
$r_{1,2} = 10\%$	$r_{0,2} =$	$r_{\overline{2}} =$
$r_{2,3} = 15\%$	$r_{0,3} =$	$r_{\overline{3}} =$

What are the three holding rates exactly?

What are the three annualized interest rates exactly?

A 1-year bond has an annualized rate of return of 5% per year. A 2-year bond has an annualized rate of return of 10% per year. A 3-year bond has an annualized rate of return of 15% per year. **First w/o a calculator**, then with: what are the three (total) holding rates of return?

First w/o a calculator, then with: what are the three annual rates of return?

Summary

Holding	Annualized	Spot/Forward
$r_{0,1} =$	$r_{\overline{1}} =$	$r_{0,1} =$
$r_{0,2} =$	$r_{\overline{2}} =$	$r_{1,2} =$
$r_{0,3} =$	$r_{\overline{3}} =$	$r_{2,3} =$

When you work with the yield curve, use over-the-envelope intuition to know the order of magnitude of your answers before you calculate them.

- Because the annualized yield is an average of spot/forward rates, the forward rates rises/declines faster than the yield curve. For example, if $r_{\overline{1}} = 5\%$ and $r_{\overline{2}} = 6\%$, then $r_{1,2} > 6\%$, because 5% and $r_{1,2}$ "geo-averaged" must come to 6%. By this argument, $r_{1,2}$ should be about 7%.

Given a full set of annualized interest rates, $r_{\overline{1}}$, $r_{\overline{2}}$, ..., $r_{\overline{T}}$, can you compute each and every T-year holding rates of return?

Given a full set of annualized interest rates, $r_{\overline{1}}$, $r_{\overline{2}}$, ..., $r_{\overline{T}}$, can you compute each and every forward rate (implied future spot rate) from $T-x$ years to T years?

Does the “yield curve” imply a unique set of forward/spot rates?

Does the complete set of spot holding rates of returns imply a unique yield curve?

Does the complete set of spot prices on zero (or other) bonds imply a unique yield curve?

Does the complete set of forward rates (plus 1-year spot rate) imply a unique yield curve?

Summary

The "yield curve" or "term structure of interest rates" is the curve plotting the spot (i.e., annualized) interest rate on the y-**axis against the time of the payment on the** x-**axis. It implies all forward interest rates.**

Look up Today's Yield Curve in the WSJ.

- **Nerd note**: Although we pretend that the WSJ quotes true 2-year interest rates, it actually quotes interest rates from 2-year coupon bonds. We know that the duration for such bonds is shorter than the maturity. Usually, the difference is not big. Unless you are a bond trader, this difference can typically be ignored.

What does an upward-sloping yield curve mean for an investor?

- 4A, Higher future inflation? (not usually)
- 4B, Higher future interest rates? (not usually)
- 4C, Bargains? (not usually) **Next**
- 4D, Risk Compensation? (most likely, yes) **Next**
  - In the real world, you have a choice:
    - Lock in the future interest rates (which gives you what we calculated). These financial markets are close-to-perfect, so there is very little transaction cost to do this.
    - Take your chances: future actual interest rates may be higher or lower that the interest rates you could lock in today.
  - A "risk premium" in which risk is higher for longer-term investments (e.g., if the firm can go bankrupt or inflation may erode the value of the repayment) would imply an upward-sloping yield curve.

What happens to the value of a bond (a loan) that you already own when interest rates increase? Does loan length matter?

Here is an example of a bond promising 8%/year:

- A 30 year bond that promises 8% interest rate costs ($\$100/1.08^{30} \approx$) \$9.94 for each \$100 promise in payment.
- If the interest rate increases by 10 basis points, the price changes to \$9.67.
- The holding rate of return is $\$9.67/\$9.94 - 1 \approx -2.74\%$. For each \$100 in investment, you would have just lost \$2.74!
- For a 1-year bond, the same calculation $p_0 = \$100/1.08 \approx \$92.5926$, $p_1 = \$100/1.081 \approx \$92.507$, and $r = p_1/p_0 - 1 \approx -0.09\%$.
- For a 1-day bond, the calculation $p_0 = \$100/1.08^{1/365} \approx \$99.979$, $p_1 = \$100/1.081^{1/365} \approx \$99.9787$, and $r = p_1/p_0 - 1 \approx -0.00025\%$. In fact, a 1-day bond is practically risk-free.

Conclusion: The interest rate sensitivity of a 30-year bond is much higher than that of a 1-year (or 1-day bond).

Is a 1-day bond riskier or a 30-year bond?

If 10bp interest rate changes are equally likely for the economy-wide 30-year rate as they are for the economy-wide 1-day rate, then 30-year bonds are riskier investments. **In the real world**, short-rates changes of 10bp are more common for short (1-year) economy-wide rates than for long (30-year) economy-wide rates, but they are not so common as to negate the fact that the 30-year is riskier than the 1-year.

If we allow for uncertainty, long-term bond investors can get more return for two reasons: because of higher expected rates of returns in the future [e.g. due higher future inflation rates], or because they are earning a "risk premium" (to be discussed soon). The empirical historical evidence suggests that most of the time, the upward slope was more of a risk premium than an expectation of higher future rates.

Corporate Lessons of Non-Flat Yield Curves

- A project of x-years is not simply the same as investing in x consecutive 1-year projects. From an investment perspective, they are different animals, and can require different costs of capital.
- The fact that longer-term projects may have to offer higher rates of return (could but) need not be due to higher risk. Even default-free Treasury bond projects in the economy that are longer-term have to offer higher rates of return than default-free Treasury bond projects in the economy that are shorter term.
- (Of course, long-term projects are also often riskier [default more often], and this also helps explain why long-term projects have to offer higher rates of return.)

Duration

A project that pays \$200 in one year and \$100 in five years has a maturity of five years, the same as a "zero-" project that pays only \$300 in five years. However, the first project is clearly shorter-term. Duration is a measure of when the cash flow arrives "on average." It is in common use in the bond context, but useful for all sorts of projects.

$$\text{Duration} = \frac{1 \times \$200 + 2 \times \$0 + 3 \times \$0 + 4 \times \$0 + 5 \times \$100}{\$100 + \$0 + \$0 + \$0 + \$100} \approx 2.3$$

A common variant called Macauley duration uses the present value and not raw cash flows. It tilts more towards the front (i.e., this duration is smaller).

Appendices (Omitted)

Locking Forward Rates: Given the current yield curve, you can lock in the future interest rate today. That is, you can eliminate all uncertainty about what interest rate that you will have to pay (or that you can earn). For example, you can buy and short Treasuries to lock in a 1-year saving Treasury rate for $1 million beginning in year 3 and lasting until year 4.

Future Interest Rates vs. Forward Rates: In the real world, future interest rates can be different from forward rates. Indeed, if you lock in a, say, 10-year-ahead 1-year savings interest rate today, on average you would have earned a higher rate of return than you would have if you had purchased 1-year savings bonds in the open market. If you are dealing with bonds, you therefore may need more notation. You now will have a future 1-year spot rate in 2030 (say $r_{2030,2031}$), and a 1-year forward rate that you can lock in today (say $f_{\text{Now},2030,2031}$, which is the 1-year forward rate locked in today). Tomorrow's locked in forward rate would be $f_{\text{tomorrow},2030,2031}$. And so on. Yikes.

Continuous Compounding: If interest is paid not once per year, but every second, this is the continuously compounded interest rate. It is often used for options pricing. OK, skipped for exams.

Strips: I cheated on the exact method to compute bond prices. The common yield curve is computed from IRRs, and not even based on actual interest rates, but based on interest quotes.