27

Options and Risk Management

. . . and some other derivatives

This chapter provides a brief introduction to the most important aspects of the area of options. It covers options basics, arbitrage relationships, put-call parity, the Black-Scholes formula (and binomial option pricing), and corporate applications of option pricing ideas and methods—but all in a very condensed form. You may prefer to resort to a full book on options and derivatives if this chapter is too telegraphic for you.

Most of the concepts in the world of financial options rely on arbitrage, which is primarily a perfect-market concept. Fortunately, for large financial institutions, the market for options seems fairly close to perfect. For smaller investors, transaction costs and tax implications can play a role. In this case, the arbitrage relations discussed in this chapter hold only within the bounds defined by these market imperfections.

ANECDOTE **A Brief History of Options**

Options have been in use since Aristotle's time. The earliest known such contract was, in fact, not a financial but a real option. It was recorded by Aristotle in the story of Thales the Milesian, an ancient Greek philosopher. Believing that the upcoming olive harvest would be especially bountiful, Thales entered into agreements with the owners of all the olive oil presses in the region. In exchange for a small deposit months ahead of the harvest, Thales obtained the right to lease the presses at market prices during the harvest. As it turned out, Thales was correct about the harvest, demand for oil presses boomed, and he made a great deal of money.

Many centuries later, in 1688, Joseph de la Vega described in *Confusion de Confusiones* how options were widely traded on the Amsterdam Stock Exchange. It is likely that he actively exploited put-call parity, an arbitrage relationship between options discussed in this chapter. In the United States, options have been traded over the counter since the nineteenth century. A dedicated options market, however, was organized only in 1973. In some other countries, option trading is banned because it is considered gambling. *Wisegeek, "What Are Futures?"*

27.A Options

Base assets and contingent claims (derivatives).

Options are examples of **derivatives** (also called **contingent claims**). A derivative is an investment whose value is itself determined by the value of some other underlying base asset. For example, a $100 side bet that a Van Gogh painting—the base asset—will sell for more than $5 million at auction is an example of a contingent claim, because the bet's payoffs are derived from the value of the Van Gogh painting (the underlying base asset). Similarly, a contract that states that you will make a cash payment to me that is equal to the square of the price per barrel of oil in 2010 is a contingent claim, because it depends on the price of an underlying base asset (oil).

Voluntary contracting ⇒ both parties are better off *ex-ante*. Only one party is better off *ex-post.*

As with any other voluntary contract, both parties presumably engage in a derivatives contract because doing so makes them better off *ex-ante*. For example, your car insurance is a contingent claim that depends on the value of your car (the base asset). Ex-ante, both the insurance company and you are better off contracting to this contingent claim than either would be without the insurance contract. This does not mean that both parties expect to come out even. *On average*, your insurance company should earn a positive rate of return for offering you such a contract, which means that you should earn a negative expected rate of return. Of course, ex-post, only one of you will come out better off. If you have a bad accident, the insurance was a good deal for you and a bad deal for the insurance company. If you do not have an accident, the reverse is the case.

Call and Put Options on Stock

Call and put options are contingent claims.

➤ *Limited liability, Sect.* **??**, *Pg.***??**.

Options are perhaps the most prominent type of contingent claim. And the most prominent option is simply the choice to walk away from an unprofitable position without retaining any obligation. A **call option** gives its holder the right to "call" (i.e., to buy) an underlying base security for a prespecified dollar amount—called the **strike price** or **exercise price**—usually for a specific period of time. A **put option** gives its holder the equivalent right to "put" (i.e., to sell) the security. Naturally, the values of these rights depend on the value of the base asset, which can fluctuate over time. Let's look at these options in more detail.

Call Options

Example 1: Call options give the right to buy—upside participation.

Exhibit 27.1 shows a number of options that were trading on May 31, 2002. For example, you could have purchased a July IBM stock call option with a strike price of $85, thereby giving you the right to purchase one share of IBM stock at the price of $85 anywhere between May 31 and July 20, 2002. Call options increase in value as the underlying stock appreciates and decrease in value as the underlying stock depreciates. If on July 20, 2002, the price of a share of IBM stock was below $85, your right would have been worthless: Shares would have been cheaper to purchase on the open market. (Indeed, exercising would have lost money: Purchasing shares that are worth, say, $70, for $85 would not be a brilliant idea.) Again, the beauty of owning a call option is that you can just walk away. However, if on July 20, 2002, the price of a share of IBM stock was above $85, then your call option (purchase right) would have been worth the difference

Underlying Base Asset S_t		Expiration T	Strike Price K	Call Price C_t	Put Price P_t
IBM	$80.50	July 20, 2002	$85	$1.900	$6.200
		Different Strike Prices			
IBM	$80.50	July 20, 2002	$75	$7.400	$1.725
IBM	$80.50	July 20, 2002	$80	$4.150	$3.400
IBM	$80.50	July 20, 2002	$90	$0.725	$10.100
		Different Expiration Dates			
IBM	$80.50	Oct. 19, 2002	$85	$4.550	$8.700
IBM	$80.50	Jan. 18, 2003	$85	$6.550	$10.200

Exhibit 27.1: *Some IBM Option Prices on t = May 31, 2002.* The original source of these prices was OptionMetrics. The expiration date T, July 20, 2002, was 0.1333 years away. (IBM's closing price at 4:00 pm EST was 5 cents lower than what the website reported.) The prevailing interest rates were 1.77% over 1 month, and 1.95% over 6 months. For up-to-date option prices on IBM options, see, for example, http://finance.yahoo.com/q/op?s=IBM, or optionmetrics.com.

between what IBM stock was trading for and your exercise price of $85. You should have exercised the right to purchase the share at $85 from the call writer. For example, if the price of IBM stock turned out to be $100, you would have enjoyed an immediate net payoff of $100-$85=$15. The relationship between the call value and the stock value at the instant when the call option expires

$$
\begin{aligned}
C_t(K = \$85, t = T) &= \max(0, S_T - \$85) \\
C_T(K, T) &= \max(0, S_T - K)
\end{aligned}
$$

where C_T is our notation for the value of the call option on the final date T, given the (pre-agreed) strike price K. If the stock price at expiration, S_T, is above K, the option owner earns the difference between S_T and K. If S_T is below K, then the option owner will not exercise the option and earn zero. (The max function means "take whichever of its arguments is the bigger.") Note that, like other derivatives, an option is like a side bet between two outside observers of the stock price. Neither party necessarily needs to own any stock. Therefore, because the person owning the call is paid $\max(0, S_T - K)$ at the final date (relative to not owning the call), the person having sold the call must pay $\max(0, S_T - K)$ (relative to not having written the call).

The upfront price of the option compensates the option writer.

Why would someone sell ("write") an option? The answer is "for the money up front." Exhibit 27.1 shows that on May 31, 2002 (when IBM stock was trading for \$80.50), an IBM call with a strike price of \$85 and an expiration date of July 20, 2002, cost \$1.90. As long as the upfront price is fair—and many option markets tend to be close to perfect—neither the purchaser nor the seller comes out for the worse. Indeed, as already noted, because both parties voluntarily engage in the contract, they should both be better off ex-ante. Of course, ex-post, the financial contract will force one side to pay the other, making one side financially worse off and the other side financially better off, relative to not having written the contract.

What could be the participants' deeper motives?

Call options are often used by shareholders to sell off some of the upside. For example, the following are common motivations for participants:

The buyer: Why would someone want to purchase a call option? It's just another way to speculate that IBM's stock price will go up—and it is very efficient in terms of its use of cash up front. In May 2002, the option to purchase IBM at \$90 until July 20, 2002, cost only \$0.725 per share, much less than the \$80.50 that one IBM share cost at the time.

The seller: As a large IBM share owner, you may have decided that you wanted to keep the upside until \$90 but did not care as much about the upside beyond \$90 (or you believed that the IBM share price would not rise beyond \$90 by July 20, 2002). In this case, you might have sold a \$90 call option now. This would have given you an immediate payment of \$0.725. You could have invested this anywhere (including into more IBM shares or Treasuries). The extra cash of \$0.725 would have boosted your rate of return if the IBM stock price had remained below \$90. But if IBM had ended up at \$120, you would have participated only in the first \$9.50 gain (from \$80.50 to \$90). (Of course, you would also have kept the upfront option payment.) The remaining \$30 of the IBM upside would have gone to your call option purchaser instead of to you.

Do you want to be caught naked?

If you write an option on a stock that you are holding, it is called "writing a covered option." Effectively, this is like a hedged position, being long in the stock and short in the call. Thus, if properly arranged, its risk is modest. However, there are also some sellers that write options without owning the underlying stock. This is called naked option writing. (I kid you not.) Lacking the long leg of the hedge, this can be a very risky proposition. In our extreme \$120 example, the option buyer would have had a rate of return on the option alone of $(\$30-\$0.725)/\$0.725 \approx 4,038\%$. Thus, the option seller would have lost 4,038%. (You can exceed –100% because your liability is not limited to your investment.) Writing naked out-of-the-money options is sometimes compared to picking up pennies in front of a steamroller—profitable most of the time, but with a huge risk.

Put Options

Example 2: Put options give the right to sell—downside protection.

In some sense, a **put option** is the flip side of a call option. It gives the owner the right (but not the obligation) to "put" (i.e., sell) an underlying security for a specific period of time in exchange for a prespecified price. For example, again in May 2002, you could have purchased a put option for the right to sell one share of IBM stock at the price of \$75 up until July 20, 2002. This option would have cost you \$1.725, according to Exhibit 27.1. Unlike a call option, a put option speculates that the underlying security will decline in value. If the price of a share of IBM stock had remained above \$75 before July 20, 2002, the put right would have been worthless: Shares could be sold for more on the open market. However, if the price of a share of IBM stock was below \$75 on the expiration date, the put right would have been worth the difference between \$75 and IBM's stock price. For example, if the IBM share price had been \$50, the put owner could have purchased one share of IBM at \$50 on the open market and exercised the right to sell the share at \$75 to the option writer for an immediate net payoff of \$25. The relationship between the put value and the stock value at the final moment when the put option expires (i.e., in this case at the end of the day on July 20, 2002) is

$$P_t(K = \$75, t = T) = \max(0, \$75 - S_T)$$

$$P_T(K, T) = \max(0, K - S_T)$$

A common use of a put is protection (insurance).

Put options are often purchased as "insurance" by investors. For example, if you had owned a lot of IBM shares when they were trading at \$80.50/share on May 31, 2002, you may have been willing to live with a little bit of loss, but not a lot. In this case, you might have purchased put options with a strike price of \$75. If IBM were to have ended up at \$60 per share on July 20, 2002, the gain on your put option (\$15/put) would have made up for some of the losses (\$20.50/share) on your underlying IBM shares. Of course, buying this put option insurance would have cost you money—\$1.725 per share to be exact.

Q 27.1. How is owning a call option the same as selling a put option? How is it different?

More Institutional Stock Option Arrangements

American options can be exercised *before* expiration. European options can be exercised only *at* expiration.

There are a variety of other option contract features. One common feature is based on the time at which exercise can occur. An **American option** allows the holder of the option to exercise the right any time up to, and including, the expiration date. The largest financial market for trading options on stocks is the **Chicago Board Options Exchange** (**CBOE**) and its options are usually of the American type. A less common form is called a **European option**. It allows the holder of the option to exercise the right only at the expiration date. The popular S&P index options are of the European type.

Splits and dividends?

What happens to the value of a CBOE stock option when the underlying stock pays a dividend or executes a stock split? In a stock split, a company decides to change the meaning, but not the value, of its shares. For example, in a 2-for-1 split, an owner who held 1,000 shares at \$80.50/share would now own 2,000 shares at \$40.25 per share (at least in a perfect market). Splitting itself should not create shareholder value—it should not change the market capitalization of the underlying company.

➤ *Stock splits, Sect.* **??**, *Pg.* **??**.

ANECDOTE **Geography and Options**

The origin of the terms “European” and “American” is a historical coincidence, not a reflection of what kind of options are traded where. Although no one seems to remember the origins of these designations, one conjecture is that contracts called “primes” were traded in France. These could only be exercised at maturity—but they were not exactly what are now called European options. Instead, the option owner either exercised (and received S-K) or did not exercise and paid a “penalty” fee of D called a “dont” (not “don’t”). There was no upfront cost. (The best strategy for the prime owner was to exercise if S - X > -D.) Because these contracts could only be exercised at maturity and because American options could be exercised at any time, the terminology may have stuck.

Incidentally, “Bermuda options,” or “Atlantic options,” can be exercised periodically before maturity but not at any other time. They are so named not because they are used in Bermuda, but because Bermuda (and of course the Atlantic Ocean) lies between Europe and America. *Jonathan Ingersoll, Yale*

Most options are adjusted for splits.

Although such a split should make little difference to the owners of the shares ($80,500 worth of shares, no matter what), it could be bad news for the owner of a call option. After all, a call with a strike price of $75 would have been **in-the-money** (i.e., the underlying share price of $80.50 was above the strike price) before the split. If the option were American, the call would be worth $5.50 per share if exercised immediately. After the split, however, the call would be far **out-of-the-money** (i.e., the underlying share price of $40.25 would be far below the strike price of $75). Fortunately, the option contracts that are traded on most exchanges (e.g., the CBOE) automatically adjust for stock splits, so that the value of the option does not change when a stock split occurs: In this case, the option’s effective strike price would automatically halve from $75 to $37.50 and the number of calls would automatically double from 1 to 2. (Completing the options terminology, not surprisingly, **at-the-money** means that the share price and the strike price are about equal.)

But options are usually not adjusted for dividends.

➤ *Dividend ex-day price drop, Sect. ??, Pg.??.*

But common options are typically *not* adjusted for dividend payments: If the $80.50 IBM share were to pay out $40 in dividends, and unless dividends fall like manna from heaven, then the post-dividend share price would have to drop to around $40.50. Therefore, the in-the-money call option would become an out-of-the-money call option. Consequently, if your call was American, you might decide to exercise your call with a $75 strike price to net $5.50 just before the dividend date.

IMPORTANT

When you purchase/value a typical financial stock option, the contract is written in a way that renders stock splits but not dividend payments irrelevant.

One option contract is (a bundle of) 100 options.

There are other important institutional details that you should know if you want to trade options. First, because the value of options can be very small (e.g., 72.5 cents for each IBM call option), they are usually traded in bundles of 100. This is called an **option contract**. Five option contracts on IBM are therefore 500 options (options on 500 shares), which in the example would cost $\$0.725 \cdot 500 = \$362.50$. Second, CBOE options typically expire on the Saturday following the third Friday of each month, which is where our 20th of July came from. Third, published option prices can be mismatched to the underlying stock price. The CBOE closing price is at 4:00 pm CST (5:00 pm EST), which is 1 hour later than the closing price from the NYSE (4:00 pm EST). This sometimes leads to seeming arbitrages in printed quotes, which are not really there. Instead, what usually happens is that the underlying stock price has changed between 4:00 pm and 5:00 pm and the printed quotes do not reflect the change. (In addition, the closing price may be a recent bid or recent ask quote, rather than the price at which you could actually transact.)

Q 27.2. An option is far in-the-money and will expire tonight. How would you expect its value to change when the stock price changes?

Q 27.3. In a perfect market, would a put option holder welcome an unexpected stock split? In a perfect market, would a put option owner welcome an unexpected dividend increase?

Option Payoffs at Expiration

Payoff diagrams describe (European) options.

It is easiest to gain more intuition about an option by studying its payoff diagram (and payoff table). You have already seen these in the building and capital structure contexts. They show the value of the option as a function of the underlying base asset at the final moment before expiration. Exhibit 27.2 shows the payoff tables and payoff diagrams for a call and a put option, each with a strike price of $90. The characteristic of any option's payoff is the kink at the strike price: For the call, the value is zero below the strike price, and a +45-degree line above the strike price. For the put, the value is zero above the strike price, and a –45-degree line below the strike price.

➤ *Payoff diagrams in the building and capital structure context, Sect. **??**, Pg.**??**.*

Optional: More Complex Option Strategies

Some common complex option strategies.

Payoff diagrams can also help you understand more complex option-based strategies, which are very popular on Wall Street. Such strategies may go long and/or short in different options at the same time. They can allow you to speculate on all sorts of future developments for the stock price—for example, that the stock price will be above $60 and below $70. In many (but not all) cases, it is not clear why someone would want to engage in such strategies, except for speculation.

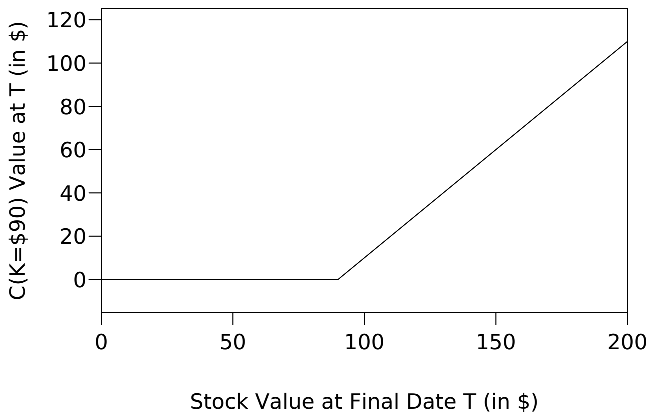
Payoff diagrams for spreads and combinations.

Two important classes of complex option strategies are **spreads**, which consist of long and short options of the same type (calls or puts), and **combinations**, which consist of options of different types.

A simple spread is a position that is long one option and short another option, on the same stock. The options here are of the same type (puts or calls) and have the same expiration date but different strike prices. For example, a simple spread may purchase one put with a strike price of $90 and sell one put with a strike price of $70. Exhibit 27.3 plots the payoff diagram for this position.

A complex spread contains multiple options, some short, others long. You will get to graph the payoff diagram of a so-called butterfly spread in Question 27.6.

Stock$_T$	Call$_T$	Put$_T$	Stock$_T$	Call$_T$	Put$_T$
\$0	\$0	\$90	\$100	\$10	\$0
\$25	\$0	\$65	\$125	\$35	\$0
\$50	\$0	\$40	\$150	\$60	\$0
\$75	\$0	\$15	\$175	\$85	\$0



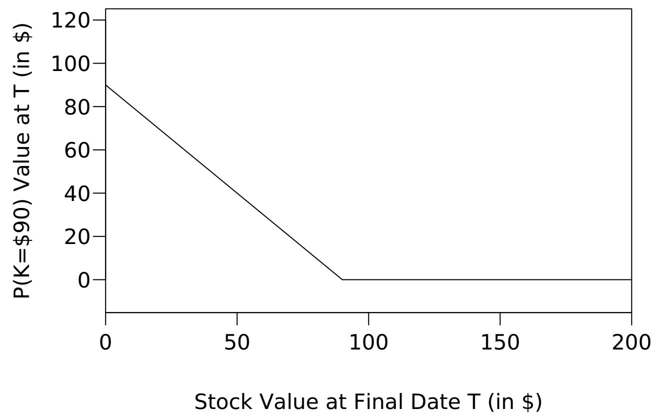


Exhibit 27.2: *Payoff Table and Payoff Diagrams of Options with Strike Price K=\$90 on the Expiration Date T.* Note: In Exhibit 27.6, we will graph the value of an option *prior to* expiration.

A straddle may be the most popular combination. It combines one put and one call, both either long or short, often with the same strike price and with the same time to expiration. You will get to graph the payoff diagram in Question 27.25.

In sum,

Option Strategy	Version A	Version B
Simple Spread	Long Call, Short Call	Long Put, Short Put
Combination	Long Call, Short Put	Short Call, Long Put
Straddle	Long Call, Long Put	Short Call, Short Put

A rarer strategy is the **calendar spread**, which is a position that is long one option and short another option, on the same stock. The options are of the same type (puts or calls) and have the same strike prices but different expiration dates. Therefore, they do not lend themselves to easy graphing via payoff diagrams because payoff diagrams hold the expiration date constant.

Payoff Table

Stock$_T$	Long Put(K=$90)	Short Put(K=$70)	Net
$50	$40	–$20	$20
$60	$30	–$10	$20
$70	$20	$0	$20
$80	$10	$0	$10
$90	$0	$0	$0
$100	$0	$0	$0

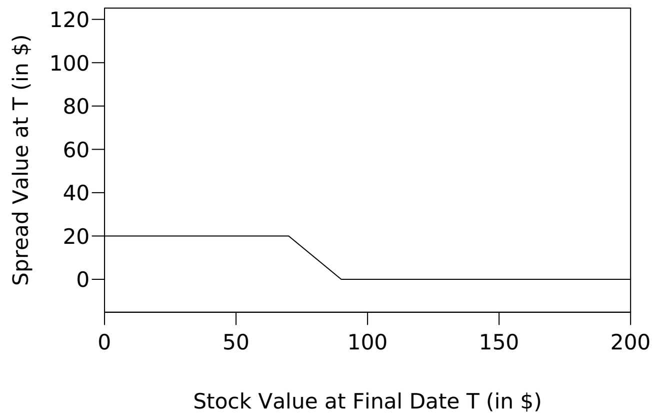


Exhibit 27.3: *Payoff Diagram of a Simple Spread.* This spread is long 1 put option with a strike price of $90 and short 1 put option with a strike price of $70.

ANECDOTE Environmental Options

Publicly traded options extend beyond stocks. For example, there is an active market in pollution options, which give option owners the legal right to spew out emissions such as CO_2. Experts generally agree that despite some shortcomings, the system of permitting trading in pollution rights and derivatives has led to a cleaner environment. It is no longer in the interest of a polluter to maximize pollution: Shutting down an old plant and selling the right to pollute can be more profitable than operating the plant.

Q 27.4. Write down the payoff table and draw the payoff diagram (both at expiration) of a portfolio consisting of 1 call option with a strike price K of $60 and 1 put option with a strike price K of $80.

Q 27.5. Write down the payoff table and draw the payoff diagram (both at expiration) of a portfolio consisting of 1 call *short* with a strike price K of $60 and 1 put *short* with a strike price K of $80.

Q 27.6. Graph the payoff diagram for the following butterfly spread:

- 1 long call option with a strike price of $50
- 2 short call options with strike prices of $55
- 1 long call option with a strike price of $60

27.B Static No-Arbitrage Relationships

There are very few pricing bounds on underlying asset prices.

How easy is it to value an underlying stock? For example, to value the shares of IBM, you have to determine all future cash flows of IBM's underlying projects with their appropriate costs of capital. You already know that this is very difficult. I cannot even tell you with great confidence that the price of an IBM share should be within a range that is bounded by a factor of 3 (say, between \$50 and \$150).

But there are good pricing bounds on their derivatives.

In contrast, it is possible to find very good pricing bounds for options. Intuitively, the law of one price works quite well for them. The reason is that you can design a clever position—consisting of the underlying stocks and bonds—that has virtually the same payoffs as a call (or a put) option. Thus, the price of the call option should be very similar to the price of the securities you need to create such a call-mimicking position. This is a no-arbitrage argument. *The price of an option should be such that no arbitrage is possible*.

➤ *Arbitrage, Sect.* **??**, *Pg.* **??**.

Some Simple No-Arbitrage Requirements

The value of the option depends on the stock, which makes the option price easier to determine.

Let us derive the first pricing bound: A call option cannot be worth more than the underlying base asset. For example, if IBM trades for \$80.50 per share, a call option with a strike price of, say, \$50 cannot cost \$85 per option. If it did, you should purchase the share and sell the call. You would make \$85-\$80.50=\$4.50 now. In the future, if the stock price goes up and the call buyer exercises, you deliver the one share you have, still having pocketed the \$4.50 net gain. If the stock price goes down and the call buyer does not exercise, you still own the share plus the upfront fee. Therefore, lack of arbitrage dictates that the value of the call C_0 now must be (weakly) below the value of the stock S_0,

$$C_0 \le S_0$$

This is an upper bound on what a call can be worth. It improves your knowledge of what a reasonable price for a call can be. It may be weak, but at least it exists—there is no comparable upper bound on the value of the underlying stock!

Selected (obvious) static no-arbitrage relations.

There are many other option pricing relations that give you other bounds on what the option price can be. For notation, call $C_0(K,t)$ the call option price now, K the strike price, (lowercase) t the time to option expiration, and P_0 the put option price now. Here are some more pricing bounds:

- Because the option owner only exercises it if it is in-the-money, an option must have a nonnegative value. Therefore,

$$C_0 \ge 0, \quad P_0 \ge 0$$

- It is better to own a call option with a lower exercise price. Therefore,

$$K_{High} \ge K_{Low} \iff C_0(K_{Low}) \ge C_0(K_{High})$$

- It is better to own a put option with a higher exercise price. Therefore,

$$K_{High} \geq K_{Low} \iff P_0(K_{Low}) \leq P_0(K_{High})$$

American options, which can immediately be exercised, enjoy further arbitrage bounds:

- The value of an American call now must be no less than what you can receive from exercising it immediately. Therefore,

$$C_0 \geq \max(0, S_0 - K)$$

- The value of an American put now must be no less than what you can receive from exercising it immediately. Therefore,

$$P_0 \geq \max(0, K - S_0)$$

- It is better to have an American call option that expires later. Therefore,

$$t_{Longer} \geq t_{Shorter} \iff C_0(t_{Longer}) \geq C_0(t_{Shorter})$$

- It is better to have an American put option that expires later. Therefore,

$$t_{Longer} \geq t_{Shorter} \iff P_0(t_{Longer}) \geq P_0(t_{Shorter})$$

These are commonly called **no-arbitrage relationships**, for obvious reasons.

Put-Call Parity

Put-call parity via example.

There is one especially interesting and important no-arbitrage relationship, called **put-call parity**. It relates the price of a European call to the price of its equivalent European put, the underlying stock price, and the interest rate. Here is how it works. Assume the following:

- The interest rate is 10% per year.
- The current stock price S_0 is \$80.
- A 1-year European call option with a strike price of \$100 costs $C_0(K{=}\$100){=}\$30$.
- A 1-year European put option with a strike price of \$100 costs $P_0(K = \$100) =$ \$50.

Further, assume that there are no dividends (which is important). Because the options are European, you only need to consider what you pay now and what will happen at expiration T. (Nothing can happen in between.) If this were the situation, could you get rich? Try the position in Exhibit 27.4. (You can check the sign, because any position that gives you a positive inflow now must give you a negative outflow tomorrow, or vice versa. Otherwise, you would have a security that always makes, or always loses, money.)

Now			At Final Expiration Time T				
		Covering S_T Range:	S_T<\$100		S_T=\$100	S_T>\$100	
Execute	**Cash Flow**	**Price S_T is:**	**\$80**	**\$90**	**\$100**	**\$110**	**\$120**
Purchase 1 call	-\$30.00	You can exercise	\$0	\$0	\$0	+\$10	+\$20
strike price K=\$100	$-\ C_0(K)$		\$0	\$0	S_T-K	S_T-K	S_T-K
Sell 1 put	+\$50.00	Your buyer can exercise	-\$20	-\$10	\$0	\$0	\$0
strike price K=\$100	$+\ P_0(K)$		S_T-K	S_T-K	S_T-K	\$0	\$0
Sell 1 share:	+\$80.00	The short is closed	-\$80	-\$90	-\$100	-\$110	-\$120
(= short 1 share)	$+\ S_0$		$-S_T$	$-S_T$	$-S_T$	$-S_T$	$-S_T$
Save money, to pay	-\$90.91	You get your money back	+\$100	+\$100	+\$100	+\$100	+\$100
the PV(strike price)	$-\ PV_0(K)$		+\$K	+\$K	+\$K	+\$K	+\$K
Net =	+\$9.09	Net =	\$0	\$0	\$0	\$0	\$0
			\$0	\$0	\$0	\$0	\$0

Exhibit 27.4: *Sample Put-Call Parity Violation.* The net arbitrage profit is (–\$30) + (+\$50) + (+\$80) + (–\$90.91) = (+\$9.09). Because $-C_0(K) + P_0(K) + S_0 - PV_0(K)$ is not \$0, this is a put-call parity arbitrage violation.

The prices in the table violate put-call parity ⇒ arbitrage.

Exhibit 27.4 shows that you could sell one put for \$50 and short one share (for proceeds of \$80 from the buyer). You would use the \$130 in cash to buy one call for \$30 and deposit \$90.91 in the bank. This leaves you with your free lunch of \$9.09. The table also shows that regardless of how the stock price turns out, you will not have to pay anything. This is an arbitrage. Naturally, you should not expect this to happen in the real world: One of the securities is obviously mispriced here. Given that the risk-free interest rate applies to all securities, and given that the stock price is what it is, you can think of put-call parity as relating the price of the call option to the price of the put option, and vice versa—and in this example, either the call is too cheap or the put is too expensive.

Put-call parity via algebra.

As usual, the algebraic formulas are just under the numerical calculations. The table shows that put-call parity means that the world is sane only if

$$-\,C_0(K) + P_0(K) + S_0 - PV_0(K) \;=\; 0 \quad\Leftrightarrow\quad C_0(K) \;=\; P_0(K) + S_0 - PV_0(K)$$

➤ *Option Prices in 2002, Exhibit* 27.1, *Pg.317*.

Let's apply put-call parity to the option prices in Exhibit 27.1. An IBM put with a strike price of \$85, expiring on July 20, 2002, costs \$6.200. The expiration was 34 out of 255 trading days away ($34/255 \approx 0.1333$ years), or, if you prefer, 50 out of 365 actual days ($50/365 \approx 0.137$ years)—this is rounding error that makes little difference. The prevailing interest rate was 1.77% per annum. Thus, the strike price of \$85 was worth $\$85/(1+1.77\%)^{0.133} \approx \$84.80$. Put-call parity implies that the call should cost

$$
\begin{array}{rcl}
C_0(K) & \approx & \$6.20 + \$80.50 - \$84.80 = \$1.90 \\
C_0(K) & = & P_0(K) + S_0 - PV_0(K)
\end{array}
$$

This was indeed the call price in the market, as you can see in Exhibit 27.1.

IMPORTANT

Given an interest rate and the current stock price, the prices of a European call option and a European put option with identical expiration dates and strike prices are related by put-call parity,

$$C_0(K) = P_0(K) + S_0 - PV_0(K) \tag{27.1}$$

The stock must not pay dividends before expiration.

Q 27.7. Write down the put-call parity formula, preferably without referring back to the text. What are the inputs?

Q 27.8. A 1-year call option with a strike price of $80 costs $20. A share costs $70. The interest rate is 10% per year.

1. What should a 1-year put option with a strike price of $80 trade for?
2. How could you earn money if the put option with a strike price of $80 traded in the market for $25 per share instead? Be explicit in what you would have to short (sell) and what you would have to long (buy).

The American Early Exercise Feature

Assuming no dividends on the stock, put-call parity implies that an American call is never exercised early.

Although put-call parity applies *only* to European options, it has the interesting and clever implication that American call options should never be exercised early. (Again, keep in mind that the underlying stock must *not* pay dividends.) Here is why: If an American call option is exercised immediately, it pays $C_0 = S_0 - K$. If the call is not exercised immediately, is the live option price more or less than this? Well, you know that the American option cannot be worth less than an equivalent European, because you can always hold onto the American option until expiration:

$$\text{American Call Value} \geq \text{European Call Value}$$

Put-call parity tells you that the European call price is

$$\text{European Call Value} = C_0 = P_0(K) + S_0 - PV_0(K)$$

$P_0(K)$ is a positive number and $PV_0(K)$ is less than K, which means that

$$\begin{aligned}\text{American Call Value} \geq \quad & \text{European Call Value} \\ & = P_0(K) + S_0 - PV_0(K) \\ & \geq S_0 - PV_0(K) \\ & \geq S_0 - K\end{aligned}$$

Therefore, the prevailing value of a live, unexercised American call is always at least equal to what you could get from its immediate exercise ($S_0 - K$). If you need money, sell the call in the market (at its arbitrage-determined value) and don't exercise it! By the way, you can also see from Exhibit 27.1 that the American call price was higher than what you could have gotten from immediate exercise. For example, the July 20, 2002, call with a strike price of $75 would have netted you only $80.50-$75=$5.50 upon immediate exercise, but $7.40 in the open market.

➤ *Option Prices in 2002, Exhibit* 27.1, *Pg.317*.

Thus, such an American call is like a European call.

In sum, the value of the right to exercise early an American call option on a non-dividend-paying stock is zero. Therefore, an American call option—even though it can be exercised before expiration—is not worth more than the equivalent European call option:

$$\text{American Call Value} = \text{European Call Value}$$

IMPORTANT

Assuming that the underlying stock pays no dividends, put-call parity implies that the value of an American call option is higher alive than if it is immediately exercised. Therefore, the American right to exercise early is worthless, and the price of a European call option is the same as the price of an American call option.

However, you may want to exercise other American options early.

However, there are cases when early exercise can be valuable, and in this case, American options are worth more than European options. Consider extreme examples for two cases:

Calls on dividend-paying stocks: If the underlying stock pays a liquidating dividend, and the call is in-the-money, it definitely becomes worthwhile for the American call option holder to exercise the call just before the dividend is paid.

Put options: If you have a 100-year put option with a strike price of $100 on a stock that trades for $1 now, it is worthwhile to exercise the option, collect the $99, and invest this money elsewhere to earn interest. Given that stocks appreciate on average, waiting 100 years to expiration reduces your payoff.

Q 27.9. Under what conditions can a European option be worth as much as the equivalent American option?

Q 27.10. Compare the direct value of exercising an American put that is in-the-money (you get $K - S_0$) to the value of the put in the put-call parity formula $P_0(K) = C_0(K) + [PV_0(K) - S_0]$. Under what conditions is it better not to exercise the American put?

27.C Valuing Options from Underlying Stock Prices

Without the put price, put-call parity does not give you the call price (and vice versa).

Put-call parity gives you the value of a call option if you know the value of the equivalent put option (or vice versa). Unfortunately, if you don't know the value of either the put or the call, you cannot pin down the value of the other. To determine the price of either, you need a formula that values one of them if all you have is the underlying stock price.

Finding the call price without the put. An intuitive explanation for binomial pricing.

Valuing an option from just the underlying stock (and risk-free bonds) requires a new idea—*dynamic arbitrage*. It asks you to construct a mimicking portfolio consisting of the underlying stock and borrowed cash, so that the call option and your mimicking portfolio always change by the same amount over the next instant. In our example, IBM stock trades for $80.50. Now presume that it can either increase by 1 cent to $80.51 or decrease by 1 cent to $80.49. (This is why this method is called binomial pricing.) How much would the value of the IBM call with a strike price of $85 change? The answer turns out to be about 0.3371 cents. Thus, your mimicking portfolio would invest about $33.71\% \cdot \$80.50 \approx \$27.14$ into IBM stock. In addition, you would have to take into consideration that you may have to pay the strike price, which is essentially handled by borrowing the appropriate amount of cash. If you do this right, then the mimicking portfolio and the call option will respond to a 1-cent change over one instant in the price of underlying IBM stock in exactly the same way. The law of one price then means that the IBM call and the mimicking portfolio (consisting of IBM stock and borrowing) should cost the same amount. Unlike static arbitrage (where you can establish a position once and then wait until expiration), dynamic arbitrage does not allow you to sit back. After this first instant, you will have to change your stock and borrowings again. If IBM goes up, then you will have to establish a stock position different from the one where IBM goes down.

Why you see a formula drop from the sky.

The details of the binomial pricing method are explained in more detail in the chapter appendix. The bad news is that it is very tedious—you have to work out all possible stock price paths until expiration. The good news is that it is a mechanical method—well suited to computer programming—and that it is very flexible. It can handle all kinds of options (even American puts and dividend-paying stocks). The best news is that there is one special-case version that gives you a quick formula for the price of a European call option on a stock without dividends. It is called the Black-Scholes formula (named after Fischer Black and Myron Scholes for their 1973 article). This formula, and the dynamic arbitrage concept on which it is based, rank among the most important advances of modern finance. Its inventors were justly honored with half an economics Nobel Prize in 1997. (The other half went to Robert Merton for his set of no-arbitrage static relationships that you already learned above.) Let me show you how to use this formula.

The Black-Scholes Formula

The Black-Scholes formula is not perfect, but it works quite well in pricing real-world options.

Unlike the CAPM, which provides only modestly accurate appropriate expected rates of return, the Black-Scholes formula is usually very accurate in practice. The reason why it works so well is that it is built around an arbitrage argument—although one that requires constant dynamic trading. It turns out that, as a potential arbitrageur, you can obtain the exact same payoffs that you receive from the call if you purchase the underlying stocks and bonds in just the right proportion and trade them infinitely often. (This is explained in detail in the chapter appendix.) In other words, if the call price does not equal the same price, then you could get rich in a perfect market. In an imperfect real world, the call price can diverge a little from the Black-Scholes price, but not much beyond transaction costs. In contrast, if the CAPM formula is not satisfied, you may find some great portfolio bets—but there are usually no arbitrage opportunities.

An Example Use of the Black-Scholes Formula

The best way to understand how to use Black-Scholes is to use it once.

Although the Black-Scholes formula may look awe-inspiring, it is not as daunting as it appears at first sight. Let's use it to determine the price of a sample call option:

Stock Price Now	S_0	\$80.50
Agreed-Upon Strike Price	K	\$85.00
Time Remaining to Maturity	t	0.1333 years
Interest Rate on Risk-Free Bonds	r_F	1.77% per year
Volatility (Standard Deviation) of the Underlying Stock	σ	30% per year

Your task is to determine the Black-Scholes call value:

$$C_0(S_0 = \$80.50, K = \$85, t = 0.1333, r_F = 1.77\%, \sigma = 30\%) = ?$$

This is a good opportunity to introduce the Black-Scholes formula:

IMPORTANT

The **Black-Scholes** formula gives the value of a call option on a stock not paying dividends:

$$C_0(S_0, K, t, r_F, \sigma) = S_0 \cdot \mathcal{N}(d_1) - PV_0(K) \cdot \mathcal{N}(d_2)$$

where you compute

$$d_1 = \frac{\log_N [S_0/PV_0(K)]}{\sigma \cdot \sqrt{t}} + {}^1\!/_2 \cdot \sigma \cdot \sqrt{t}$$

and

$$d_2 = d_1 - \sigma \cdot \sqrt{t}$$

The five inputs are as follows:

S_0	is the current stock price.
t	is the time left to maturity.
K	is the strike price.
$PV_0(K)$	is the present value of K that depends on r_F (the risk-free interest rate input, which is used only to compute $PV_0(K)$).
σ	is the standard deviation of the underlying stock's continuously compounded rate of return, and it is often casually called just "the stock volatility."

The σ is very similar to the standard deviation, $\mathcal{S}dv$ (from Chapter **??**), of the stock's rate of return. But in contrast to an ordinary rate of return standard deviation, each rate of return must first be converted into its continuously compounded equivalent rate of return (from Section App.5.F on 21). You can do this by calculating the natural log of one plus the rate of return for each value. For example, if the two simple rates of return are +1% and –0.5%, you would compute the standard deviation from $\log_N(1+1\%) \approx 0.995\%$ and $\log_N(1-0.5\%) \approx -0.501\%$. The returns (and therefore $\mathscr{S}dv$ and σ) are similar if rates of return are low.

Note that the three parameters t, r_F, and σ have to be quoted in the same time units. (Typically, they are quoted in annualized terms.) These are the two functions:

$\log_N(\cdot)$	is the *natural* log.
$\mathcal{N}(\cdot)$	is the **cumulative normal distribution** function. (Spreadsheets call this the "**normsdist()**" function.) You can also look up its values in a table in the book appendix on **??**.

This requires five steps:

1. Compute the present value of the strike price. For the approximately 7 weeks left, the interest rate would have been $(1+1.77\%)^{0.1333}-1 \approx 0.2342\%$. Therefore, the $PV_0(\$85) \approx \$84.80$.
2. Compute the input d_1, which is needed later as the argument in the left cumulative normal distribution function:

$$
\begin{aligned}
d_1 &= \frac{\log_N[S_0/PV_0(K)]}{\sigma\cdot\sqrt{t}} + 1/2\cdot\sigma\cdot\sqrt{t} \\
&= \frac{\log_N[\$80.50/PV_0(\$85)]}{30\%\cdot\sqrt{0.1333}} + 1/2\cdot 30\%\cdot\sqrt{0.1333} \\
&\approx \frac{\log_N(\$80.50/\$84.80)}{30\%\cdot 0.365} + 1/2\cdot 30\%\cdot 0.365 \\
&\approx \frac{\log_N(0.949)}{10.95\%} + 1/2\cdot 10.95\% \\
&\approx \frac{-0.052}{10.95\%} + 5.48\% \\
&\approx -47.52\% + 5.48\% \\
&\approx -42.04\%
\end{aligned}
$$

(My calculations could be a little different from yours because I am carrying full precision.)

3. Compute d_2, the argument in the right cumulative normal distribution function:

$$
\begin{aligned}
d_2 &= d_1 - \sigma\cdot\sqrt{t} \\
&\approx -42.04\% - 30\%\cdot\sqrt{0.1333} \\
&\approx -42.04\% - 10.95\%
\end{aligned}
$$

$$\approx \; -53.00\%$$

4. Look up the standard normal distribution for the d_1 and d_2 arguments in Exhibit **??**, or use the spreadsheet **normsdist()** function:

➤ *Cumulative normal distribution probabilities, Exhibit **??**, Pg.**??**.*

$$\mathcal{N}(-0.4204) \approx 0.3371, \mathcal{N}(-0.5300) \approx 0.2981$$

5. Compute the Black-Scholes value, $C_0(S_0 = \$80.50, K = \$85, t = 0.1333, r_F = 1.77\%, \sigma = 30\%)$:

$$\begin{array}{lccc} & S_0 \cdot \mathcal{N}(d_1) & - & PV_0(K) \cdot \mathcal{N}(d_2) \\ \approx & \$80.50 \cdot \mathcal{N}(-0.4204) & - & \$84.80 \cdot \mathcal{N}(-0.5300) \\ \approx & \$80.50 \cdot 0.3371 & - & \$84.80 \cdot 0.2981 \\ \approx & \$27.14 & - & \$25.28 \\ \approx & \$1.86 & & \end{array}$$

   Let me also note that if you want to hedge your option with stock, $\mathcal{N}(d_1)$ is the amount of stock that you need to purchase to be as exposed as the option to changes in the underlying stock. It is called the **hedge ratio**. In this example, you would have to purchase $27.14 worth of stock.

In sum, a call option with a strike price of $85 and 0.1333 years left to expiration on a stock with a current price of $80.50 should cost about $1.86, assuming that the underlying volatility is 30% per annum and the risk-free interest rate is 1.77% per annum. Trust me when I state that the empirical evidence suggests that 30% per annum was a reasonably good estimate of IBM's volatility in 2002. If you look at Exhibit 27.1, you will see that the actual call option price of just such an option was $1.90, not far off from the theoretical Black-Scholes value of $1.86.

➤ *Option prices in 2002, Exhibit 27.1, Pg.317.*

Q 27.11. What is the value of a call option with infinite time to maturity and a strike price of $0? Use the parameters of the example: S_0=$80.50, r_F =1.77%, and $\sigma = 50\%$.

Q 27.12. Price a call option with a stock price of $80, a strike price of $75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of 20% on the underlying stock.

The Black-Scholes Value for Other Options

B-S can be used to price certain types of options, but not others.

The Black-Scholes formula prices European call options for stocks that pay no dividends. How can you apply the Black-Scholes formula to other options? First, the good news:

American calls on stocks without dividends: Because you would never exercise such a call before expiration, the value of an American call is equal to the value of a European call. Therefore, the Black-Scholes formula prices such American call options just as well as European call options.

➤ *No early exercise, Sect.* 27.B, *Pg.327*.

European puts: If you know the value of the European call option, you can use put-call parity to determine the value of a *European* put option with the same strike price and maturity as the call option. In our example,

➤ *Put-call parity, Formula* 27.1, *Pg.327*.

$$
\begin{aligned}
P_0 &\approx \$1.86 - \$80.50 + \$84.80 = \$6.16 \\
P_0 &= C_0 - S_0 + PV_0(K)
\end{aligned}
$$

This happens to be close to, but not exactly equal to, the real-world (though American) put price of \$6.20 in Exhibit 27.1.

Now the bad news: For other options, although there are sometimes ways to bend the Black-Scholes formula, you generally have to use the more complex binomial valuation technique explained in the chapter appendix to get an exact solution. This applies to American calls on dividend-paying stocks and to American puts.

Q 27.13. Price an IBM put option with a strike price of \$100, using the parameters of the example in the text: $t=0.1333$, $r_F=1.77\%$, $\sigma = 30\%$, $S_0=\$80.50$.

1. What is the price if the option is European?
2. What is the price if the option is American? Would you continue holding onto it?

Synthetic Securities

How can you make a put option yourself?

A different way to look at arbitrage relationships is to recognize that they define securities. That is, even if a put option were not available in the financial markets, it would be easy for you to manufacture one (assuming minimal transaction costs, of course). For example, return to the put-call parity relationship. It states that European options have the relationship

$$C_0(K) = P_0(K) + S_0 - PV_0(K) \iff P_0(K) = C_0(K) - S_0 + PV_0(K)$$

Instead of purchasing one put option, you can purchase one call option, short one stock, and invest the present value of the strike price in Treasuries. You would receive the same payoffs as if you had purchased the put option itself. Therefore, you have manufactured a synthetic put option for yourself.

Making and selling synthetic securities is big business on Wall Street.

Creating synthetic securities has become a big business for Wall Street. For example, a client company owning gas stations may wish to obtain an option to purchase 10,000 barrels of crude oil in 10 years at a price of $50 per barrel. A Wall Street supplier of such call options models the price of oil and determines the appropriate value of a synthetic call option. The Wall Street supplier then sells the call option to the gas stations for a little more. But would the Wall Street firm now not be exposed to changes in the oil price? Yes—but it would in turn try to hedge this risk away. In this example, the Wall Street firm could undertake a (usually dynamic) hedge—the same idea that underlies the Black-Scholes formula. That is, it would first determine its hedge ratio, which is the amount by which the value of a synthetic 10-year call option with a strike price of $50 per barrel changes with the underlying oil price now. Say this value is 0.08. In this case, the Wall Street firm would purchase a forward contract for $10,000 \cdot 0.08 = 800$ barrels of oil. If the price of oil increases, then the Wall Street firm's own position in oil increases by the same amount as its obligation to the gas station company. This way, the Wall Street firm has low or no exposure to changes in the underlying oil price. And it has added value to its clients through its better ability to execute and monitor such dynamic hedges than the clients themselves.

➤ *Hedge ratio,* *Sect.* 27.D, *Pg.336*.

➤ *Forward contracts,* *Sect.* 26.A, *Pg.282*.

27.D The Black-Scholes Inputs

Let us now look a bit more closely at the five ingredients of the Black-Scholes formula.

Obtaining the Black-Scholes Formula Inputs

Only σ, the standard deviation of the rate of return on the underlying stock, is difficult to estimate.

The first four inputs, S_0, K, t, and r_F, either are given by the option contract (the strike price K and time to expiration t) or can be easily found online (the current stock price S_0 and the risk-free interest rate r_F [required to compute $PV_0(K)$]). Only one input, σ, the standard deviation of the underlying stock returns, has to be guesstimated. There are two methods to do so.

1. The old-fashioned way uses, say, 3-5 years of historical stock returns and computes the standard deviation of daily rates of return:

$$\sigma_{\text{Daily}} = \sqrt{\frac{\text{Sum from Day 1 to N: } (r_t - \bar{r})^2}{N - 1}}$$

   (To be perfectly accurate, the rates of return that you should be using here are continuously compounded, not simple rates of return.) Then, this number is annualized by multiplying it by $\sqrt{255}$, because 255 is the approximate number of trading days. For example, if the daily standard deviation is 1%, the annual standard deviation would be $\sqrt{255} \cdot 1\% \approx 16.0\%$. (Annualization is done by multiplying a standard deviation by the square root of the number of periods.)

➤ *Continuous compounding,* *Sect.* App.5.F, *Pg.21*.

➤ *Annualization and sdv time scaling,* *Pg.??*.

2. If other call option prices are already known, it is possible to extract a volatility estimate using the Black-Scholes formula itself. For example, assume that the price of the stock is $80.50 and the price of a July call with a strike price of $80 is $4.15.

$$C_0(\,S_0 = \$80.50, K = \$80, t = 34/255, r = 1.77\%, \sigma = ?\,) \approx \$4.15$$

What is the volatility of the underlying stock that is consistent with the $4.15 price? The idea is to try different values of σ until the Black-Scholes formula exactly fits the known price of this option.

Start with a volatility guess of 0.20. After tedious calculations, you find that

$$C_0(\,S_0 = \$80.50, K = \$80, t \approx 0.1333, r = 1.77\%, \sigma = 0.20\,) \approx \$2.70$$

Option values increase with uncertainty, so this was too low a guess for σ. Try a higher value—say, 0.50:

$$C_0(\,S_0 = \$80.50, K = \$80, t \approx 0.1333, r = 1.77\%, \sigma = 0.50\,) \approx \$6.18$$

Too high. Try something in between. (Because $4.15 is closer than $2.70 than it is to $6.18, try something a little bit closer to 0.20—say, 0.25.)

$$C_0(\,S_0 = \$80.50, K = \$80, t \approx 0.1333, r = 1.77\%, \sigma = 0.25\,) \approx \$3.27$$

Too low, but pretty close already. After a few more tries, you can determine that $\sigma \approx 0.325$ is the volatility that makes the Black-Scholes option pricing value equal to the actual call option price of $4.15.

You can now work with this **implied volatility** estimate as if it were the best estimate of volatility, and use it to price other options with the Black-Scholes formula. Unlike the historical estimated volatility, the implied volatility is forward-looking! That is, it is the market guess of what volatility will be like in the future.

Obtaining an implied volatility is such a common procedure that many Web pages provide both the option price and the implied volatility. For instance, Exhibit 27.5 shows OptionMetrics' reported implied volatilities. For the specific $80 July call, OptionMetrics computed an implied volatility of 32.58%—just about the 32.5% that we computed ourselves.

Sometimes, this implied volatility is even used interchangeably with the option price itself. That is, instead of reporting the Black-Scholes call price, traders might just report that the option is priced at a "32.5% vol." This makes it sometimes easier to compare different options. Exhibit 27.5 shows that the $75 July call has a price of $7.40, while the $85 January put has a price of $10.20. How do you compare the two? Quoting them as volatilities–34.89% versus 31.40%—makes them easier to compare.

The Black-Scholes formula is not the only option pricing formula, although it is by far the most common and also usually the easiest to use. It is pretty accurate. However, there are similar formulas based on the same dynamic trading concept that can price options just a little better. In particular, they can explain what would be an anomaly from the perspective of the Black-Scholes formula: The real-world prices of options that are far out-of-the-money—both calls and puts—are typically higher than what the Black-Scholes formula suggests. Put differently, according to the Black-Scholes formula,

The volatility "smile," an empirical regularity, suggests that B-S prices for deep-out-of-the-money options are too low.

Underlying Base Asset		Expiration T	Strike Price K	Option Type	Option Price	Implied Volatility	Option Type	Option Price	Implied Volatility
IBM	$80.50	July 20, 2002	$85	Call	$1.900	30.38%	Put	$6.200	29.82%
		Different Strike Prices							
IBM	$80.50	July 20, 2002	$75	Call	$7.400	34.89%	Put	$1.725	34.51%
IBM	$80.50	July 20, 2002	$80	Call	$4.150	32.58%	Put	$3.400	31.67%
IBM	$80.50	July 20, 2002	$90	Call	$0.725	29.24%	Put	$10.100	29.18%
		Different Expiration Dates							
IBM	$80.50	Oct. 19, 2002	$85	Call	$4.550	31.32%	Put	$8.700	31.61%
IBM	$80.50	Jan. 18, 2003	$85	Call	$6.550	31.71%	Put	$10.200	31.40%

Exhibit 27.5: *Adding Implied Volatilities to Exhibit 27.1.* The source of both prices and implied volatilities was OptionMetrics on May 31, 2002. July 20 was 0.1333 years away. The prevailing interest rates were 1.77% over 1 month, and 1.95% over 6 months.

out-of-the-money options are priced as if their volatilities are higher than that of options that are at-the-money. If you draw the implied volatilities as a function of strike price, you get a so-called **volatility smile**—which is exactly what this empirical regularity is called by traders. One explanation for the smile is that there is a rare probability of a large stock price shock that is ignored by the Black-Scholes model. This may indeed be why far-out-of-the-money options are more expensive in the real world than in the model. It is especially plausible for puts, which can serve as insurance against a stock market crash, but perhaps less plausible for calls. For hardcore option traders, this opens up another question: If there is no longer just one implied volatility for a stock but different ones depending on the strike price, then which of these should you use? To predict future volatility, the recommendation here is to use at-the-money options. Historically, they have tended to predict volatility better than out-of-the-money options.

Comparative Statics for the Black-Scholes Formula

How the B-S formula changes with its inputs.

If you have solved all the exercises from the previous section (as you should have before proceeding!), you have already seen how the Black-Scholes call option value changes with its inputs. Specifically:

Current stock price (S_0)—*positive*: A call option is worth more when the stock price now is higher. This was also a static no-arbitrage relationship, and the Black-Scholes formula obviously must obey it. Furthermore, not only do you know that

➤ *Arbitrage, Pg.324.*

the Black-Scholes formula increases with S, but you can even work out by how much. Look at the Black-Scholes formula:

$$C_0(S, K, t, r_F, \sigma) = S \cdot \mathcal{N}(d_1) - PV_0(K) \cdot \mathcal{N}(d_2)$$

The stock price appears at this very high level, separate from the strike price K, and multiplied only by $\mathcal{N}(d_1)$. It turns out that $\mathcal{N}(d_1)$ is how the value of the call changes with respect to *small* value changes in the underlying stock price. For example, if $\mathcal{N}(d_1) \approx 0.3371$, then for a 10 cent increase in the value of the underlying stock, the value of the call option increases by 3.371 cents. Put differently, if your mimicking arbitrage position is long 33.71 shares and short 100 options, then your overall portfolio will not be affected one way or the other when the underlying stock price increases (or decreases) by 1 cent. You are said to be hedged against small changes in the stock price; that is, your portfolio is insured against such changes. For this reason, $\mathcal{N}(d_1)$ is also called the **hedge ratio**. Option traders also call it the **delta**. $\mathcal{N}(d_1)$ is the number of stocks that you need to purchase in order to mimic the behavior of your one option. For example, if right now the value of your call option increases by about \$0.0025 when the underlying stock price increases by \$0.01, then your hedge ratio is 0.25. If you own four of these call options, your total option position would change in value by the same $\$0.0025 \cdot 4 = \$0.01$ amount that it would change if you owned one stock. (In addition, option traders often want to know how quickly the delta [the stock position] itself changes when the underlying stock price changes. This is called the **gamma** of the option. You can think of it as the delta of the delta.)

Strike price (K)—*negative*: A call option is worth more when the strike price is lower. Again, this was also a static arbitrage relationship.

Time left to maturity (t)—*positive*: A call option is worth more when there is more time to maturity. Again, this was also a static arbitrage relationship. (The change in the price of the option as time changes is commonly called **theta**.)

Interest rate to maturity (r_F)—*positive*: A call option is worth more when the interest rate is higher. This comparative static is not as intuitive as the three previous "comparative statics." My best attempt at explaining this intuition is that as the call option purchaser, you do not need to lay out the cash to cover the strike price immediately. You live on "borrowed" money. The higher the interest rate, the more value there is to you, the call owner, not to have to pay the strike price up front.

This is most obvious when the option is far in-the-money. For example, take a 1-year option with a strike price of \$40 on a stock with a price of \$100. Assume that the volatility is zero. If the interest rate is zero, the value of the call option is \$60: With no volatility, you know that the option will pay off \$60, and with an interest rate of zero, the value of the future payoff is the same as its present value. However, if the interest rate is 20%, then you can invest the \$40 in bonds for 1 year. Therefore, the value of the option is \$60 (at exercise), plus the \$8 in interest earned along the way—a total of \$68. (The change in the price of the option as the risk-free rate changes is commonly called **rho**.)

Volatility to maturity (σ)—*positive*: A call option is worth more when there is more volatility. When the underlying stock increases in volatility, the call option holder gets all the extra upside, but does not lose more from all the extra downside (due to limited liability). This increases the value of the option. If this comparative static is not obvious, then ask yourself whether you would rather own an option with a strike price of $100 on a stock that will be worth either $99 or $101 at expiration, or on a stock that will be worth either $50 or $150 at expiration. Holding everything else constant, an option on a more volatile asset is worth more. (The change in the price of the option as volatility changes is commonly called **vega**.)

But where is the expected rate of return on the stock in the B-S formula?

There is one counterintuitive feature of the Black-Scholes formula: The expected rate of return on the underlying stock plays no role. This is because the other inputs, most of all the stock price (but also the interest rate and volatility), already incorporate the expected rate of return on the stock and therefore all the necessary information that you need to price an option. (Different purchasers can even disagree as to what the expected rate of return on the stock should be and still agree on the appropriate price on the option.)

Q 27.14. What is the delta of an option? Does it have another name, too?

Q 27.15. In words, how does the value of a call option change with the Black-Scholes inputs?

Value Prior to Expiration

The B-S value can be used to plot values prior to expiration.

➤ *Final date payoffs*, *Sect.* 27.A, *Pg.321*.

The Black-Scholes formula allows you to determine the price of a call option not only *on* the final expiration date, but also *before* the final expiration date. Exhibit 27.6 plots the Black-Scholes value of a call option with a strike price of $90, an interest rate of 5%, and a standard deviation of 20% for three different times to expiration. The figure shows that the Black-Scholes value is always strictly above $\max(0, S_0 - K)$—otherwise, you could arbitrage by purchasing the call option and exercising it immediately. Moreover, you also already know that calls must be worth more when the underlying stock value is higher and when there is more time left to expiration. The figure nicely shows all of these features.

Option Riskiness

Options with different strike prices have different risk profiles.

You can now ask another interesting question: What are the advantages and disadvantages of call options with different strike prices? The answer is that different options provide different risk profiles. For example, say the stock was trading at $100, 3 months prior to option expiration, the annual interest rate was 5%, and the annual standard deviation of the stock's underlying rates of return was 20%. According to Black-Scholes, a call option with a strike price of $50 would have cost $50.61. A call option with a strike price of $90 would have cost $11.65. And a call option with a strike price of $120 would have cost $0.20. All are fair prices. But consider what happens if the stock were

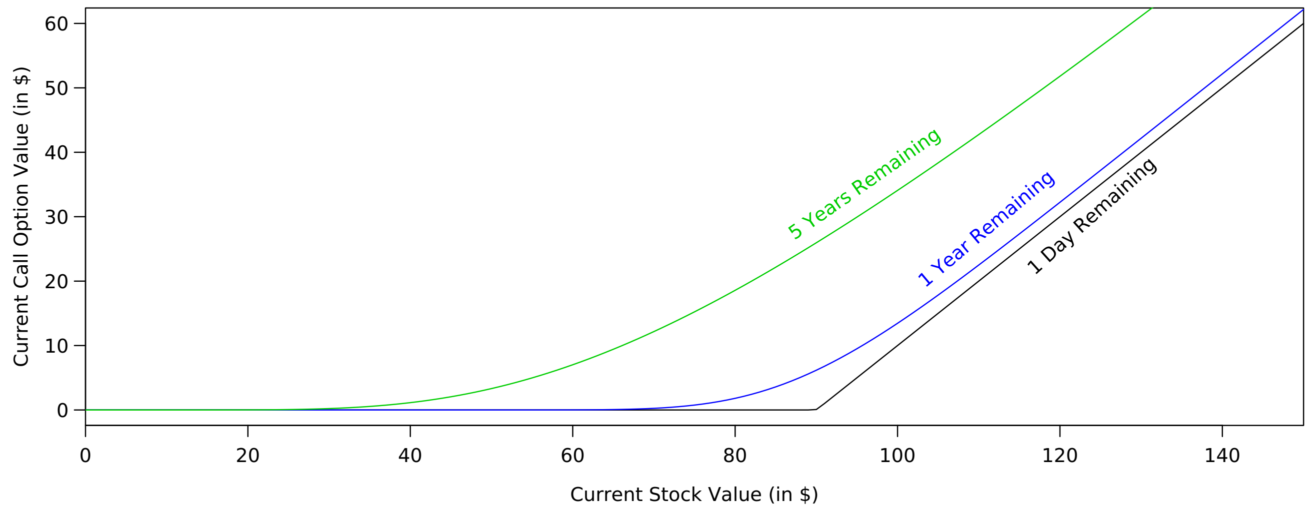


Exhibit 27.6: *Black-Scholes values prior to, and at, expiration.* In this example, the time to expiration is either 1 day, 6 months, or 5 years. In all cases, the strike price is K=$90, the annual interest rate is 5%, and the annual standard deviation is 20%.

to end up either very, very high or very, very low. If the stock price ends up at $70, the $50 option is the only one worth exercising, providing its holder with a $20 payoff. This is equivalent to a rate of return of $(\$20 - \$50.61)/\$50.61 \approx -60\%$. Exhibit 27.7 shows this calculation as well as a couple more. The call with the strike price of $50 is relatively safe compared to those with higher strike prices: It is in-the-money in both cases. The call with the strike price of $90 has roughly a 50-50 chance of losing everything—but it provides more "juice" for each dollar invested *if* it expires in-the-money. Finally, the call with the strike price of $120 is very likely to be a complete loss—but if the stock price were to exceed the strike price even by a little, the rate of return would quickly become astronomical. The rates of return on the four call options are graphed in Exhibit 27.7.

27.E Corporate Applications

Actually, the current chapter is not the first time you have encountered options. On the contrary.

Call Option Strike Price	Price Now Payoff at T	Stock Will End at $70 Payoff at T	Return	Stock Will End at $130 Payoff at T	Return
Call (Strike = $0)	$90.00	$70–$0	–22%	$130–$0	+44%
Call (Strike = $50)	$50.61	$70–$50	–60%	$130–$50	+58%
Call (Strike = $70)	$30.85	$70–$70	–100%	$130–$70	+94%
Call (Strike = $90)	$11.65	$0	–100%	$130–$90	+243%
Call (Strike = $100)	$4.60	$0	–100%	$130–$100	+552%
Call (Strike = $120)	$0.20	$0	–100%	$130–$120	+4,900%
Call (Strike = $130)	$0.02	$0	–100%	$130–$130	–100%

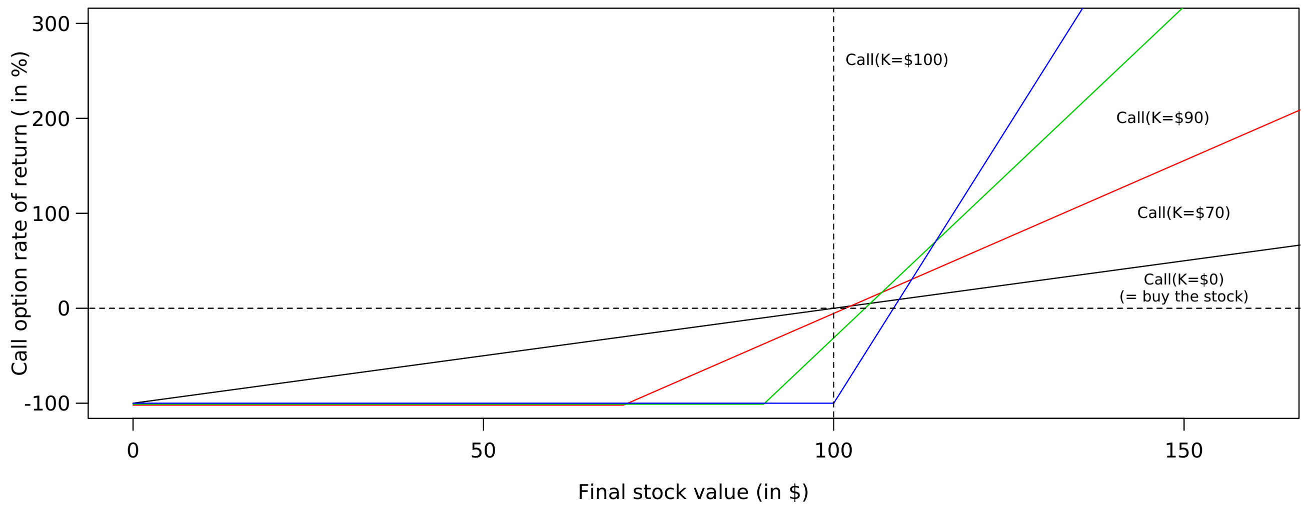


Exhibit 27.7: *Rates of Return on Call Option Investments.* In all cases, the current stock price is $100, the option is 3 months before expiration, the interest rate is 5%, and annual volatility is 20%.

Déjà Vu: Securities as Financial Options

Limited liability building ownership is a call option.

➤ *Payoff Table, Exhibit **??**, Pg.**??**.*

The first time you worked with options was when you learned about uncertainty. In Section **??**, you computed the value of levered equity ownership under limited liability. Limited liability is, at its heart, an option—the option to walk away without owing anything else.

Let's put the example from Exhibit **??** of levered equity in a building into option's lingo. If you owe a $25,000 mortgage, then your levered equity ownership is in effect a call option with a strike price of $25,000. If your building ends up being worth more than $25,000 (at loan expiration), it is in your interest to pay off the mortgage and keep the rest. If your building ends up being worth less than $25,000, you walk away and end up with $0. Alternatively, by put-call parity, you can think of equity *with* limited liability as being the same as a portfolio of equity *without* limited liability plus a put option with a strike price of $25,000, plus $25,000 in a loan. If the building ends up being worth only $20,000, you exercise the put. This means that you sell your $20,000 house and the put gives you the $25,000 - $20,000 = $5,000 profit. You use the $20,000 + $5,000 to pay off the $25,000 loan.

Levered equity (stock) is a call option.

We expanded on the building example in Section **??**. Equity holders in corporations are also limited liability owners. They are in-the-money only *after* the corporate debt is paid off. Like a building owner, a stockholder has the option to walk away without having to make up further losses to creditors. Therefore, shareholders' levered equity is essentially an option on the value of the underlying base asset, which is the firm. You can even see the equivalence of a financial option and levered equity by comparing their payoff diagrams in Exhibits **??** and 27.2, respectively. Conversely, corporate debt is like a portfolio of risk-free bonds plus a put option sold to equity owners:

➤ *Levered equity (shares), Sect.* **??**, *Pg.***??**.

- If the firm is worth a lot, the shareholders pay the face value of the bonds. This is the horizontal line in the payoff diagram.
- If the firm is worth very little, the shareholders walk away from the firm: They exercise their right to sell the firm to the creditors for the face value of the corporate debt. Creditors lose an amount that increases with the difference between the face value of the debt and the actual value of the firm. This is the diagonal line in the payoff diagram.

Option pricing techniques can help you understand and value corporate securities.

The direct-options perspective on the cash flow rights of securities can be quite useful. First, you can gain qualitative insights. For example, you know that the value of an option increases with the volatility of the underlying base asset. Therefore, levered shareholders should prefer more risky projects to less risky projects. Second, you may even be able to obtain quantitative solutions for the value of corporate securities using option pricing tools. If you can learn what process the firm's underlying value follows, you might even be able to use the Black-Scholes formula to derive an appropriate price for the firm's levered equity.

➤ *Risk-shifting, Sect.* **??**, *Pg.***??**.

Q 27.16. Is it possible to have a security that is an option on an option?

Déjà Vu: Real Projects as Options

A real option depends not on an underlying financial asset (such as a stock), but on an underlying real asset.

The second time you worked with options was when you learned how to work with "real options" in Section **??**. Recall that I explained that it is important to recognize the real options features of your projects and to value them properly. A real option is really the value of your flexibility to respond to changing environments in the future. For example, if you have the ability to shut down production if the market price of your output product were to fall, then you have an option on a base asset that is the market price of your output product. Your option's strike price would be equal to the output price at which production becomes profitable.

➤ *Real options, Sect.* **??**, *Pg.***??**.

Sometimes, you can even use the B-S formula to value a real option.

In Section **??**, we used a tree approach for valuing a number of these real options. It is almost the same approach as the binomial approach explained in this chapter's appendix. The difference is that in the tree framework of the earlier chapter, you had to provide probabilities of up and down movements, and then use standard discounting over time. In the binomial framework in this chapter, you do not have to guess the discount rate. (The underlying base asset is a traded stock. Recall also that the Black-Scholes formula does not ask you for an expected rate of return as an input.) This is a nice advantage, but not a big one. The main difficulty is writing down the tree payoffs in the first place and working out what the optimal operating policy is (as a function of different state variables). Unfortunately, compared with the wealth of options embedded in real projects and their value dependence on many underlying factors, even complex financial options seem like child's play. It is rare that you can use the same financial option tools, like the Black-Scholes formula, to value a real option—more commonly, a tree approach using CAPM -type (or even risk-neutral) discounting makes your task simpler. Fortunately, the approach to valuing real options remains conceptually very similar, so once you understand one, the other is much easier.

book.ivo-welch.info

(This book also has a complete web chapter dedicated to real options valuation. Even this dedicated chapter can only scratch the surface. Other authors have written entire books on the subject.)

Q 27.17. You have received an offer to buy a lease for 1 week's worth of production (100 ounces) in a particular gold mine. This lease will occur in exactly 18 months. It is an old mine, so it costs \$400/ounce to extract gold. Gold is trading for \$365/ounce now but has a volatility of 40% per annum. The prevailing interest rate is 10% per year. What is the value of the gold mine?

Q 27.18. Now assume that you own this mine. If the mine is inexhaustible, but can only extract 100 ounces per week, and the production cost increases by 20% per year (starting at \$400 next week, your first production period), how would you value this mine? (Do not solve this algebraically. Just think about the concepts.)

Déjà Vu: Risk Management

Risk management is the manipulation of the firm's exposure to risk.

➤ *Currency hedging*, *Sect.* 26.D, *Pg.301*.

The third time you worked with derivatives (though not with options) was in Chapter 26, which showed you how a firm can hedge its exchange rate exposure. For example, consider an American corporation that has just sold its product to a German corporation for payment in euros in 6 months but that must pay its suppliers for its own inputs in U.S. dollars. It can lock in the current dollar value of its future euro receipts by selling some euro futures. This is a form of **risk management**, the deliberate manipulation of the risk exposure that the corporation faces. (For most companies, risk management means lowering risk exposure.) Risk management is worth covering in more generality. For example, a firm may also purchase liability insurance to protect it against occasional random mishaps. Or it may want to hedge its credit risk or oil risk exposures. Options and other derivatives are natural tools that can help to manage corporate risk, which is why we cover risk management in this chapter.

Why Hedge?

Risk management is irrelevant in a perfect market.

In a perfect market, there is no value to risk management. You learned in Section **??** that if investors can freely do or undo a transaction, it cannot add value. If the firm sells euros for dollars at the appropriate price, investors can easily undo this by taking the offsetting position (buying euros). Investors' return would again be based only on the value of the unhedged firm. Equivalently, if the firm were not to hedge the currency, investors could hedge for themselves. They could sell euros, and their return would come from the unhedged firm plus the value of the hedge. This argument is really the same as the Modigliani-Miller indifference proposition in the context of capital structure. Indeed, hedging risk is often the equivalent of a capital structure activity—the company can often share its risk either by selling equity or by hedging.

➤ *Corporate consequences of market efficiency,* Sect. **??**, Pg.**??**.

➤ *Modigliani-Miller propositions,* Sect. **??**, Pg.**??**.

Risk management can be useful in an imperfect market.

It is only in an imperfect market that risk management matters. In this case, you have to think about all the capital structure issues raised in Chapters **??** and **??**:

- Can risk management change the taxes paid by the corporation or its investors?
- Can it reduce deadweight financial distress costs?
- Can it worsen or alleviate conflicts between bondholders and stockholders?
- Can it induce the manager to work harder and make better decisions, or work less and make worse decisions?

And so on. The considerations in favor of risk management are usually the same as those in favor of having more equity and less debt. For example, an airline company could avoid the financial distress that rising fuel prices could cause if it were to purchase fuel futures. If the fuel price were to rise, its flight operations would turn unprofitable, but its fuel hedge would make money. Such a fuel hedge could add value if it avoids the collapse of an otherwise valuable underlying business. But it could also subtract value if it prevents the managers (the agents of the owners) from shutting down the airline and selling its assets if this were the value-maximizing action.

How to Hedge

The firm can hedge different types of risks with many tools—not just with derivatives.

The basic idea of risk management through hedging is simple: The firm reduces a source of risk that it otherwise faces. The firm has a number of risk-management tools at its disposal:

- It can buy a policy from an insurance company that may specialize in, and thus understand and manage, the risk better. This works especially well if the risks are idiosyncratic—for example, the risk of a firm being sued or the risk of a firm's building collapsing. Insurance policies may work—but often less well—for more systematic risks, such as industry risks, commodity price risks, exchange rate risks, or interest rate risks. (In the credit crisis of 2008, investors that had purchased insurance against the credit risk in bonds suddenly learned that their main risk was not that just the underlying issuer would go out of business. Rather, it was also that many low-credit bonds could default at the same time and the insurer itself could go out of business. In other words, these investors mistook a true systematic risk for idiosyncratic risk and thus used the wrong tool [insurance policies] as protection.)

- It can execute or not execute certain projects. For example, it can take fewer projects or reduce its risk by preferentially taking more projects that have a lower correlation with its existing operations. This is the diversification intuition we used for the CAPM, except that the firm uses it here to reduce its own firm risk and not its investors' portfolio risks.
- It can buy or sell contracts in the financial markets. For example, it can buy or sell options (or futures or stocks) to shift the risk to another party. This is especially popular if the risk is systematic and economy-wide. (In some cases, both contract parties may experience a decline in risk. For example, an oil producer may want to sell the oil futures that an oil consumer would want to buy. In other cases, there may be firms that specialize in absorbing risks. [This is one of the roles of funds, especially hedge funds.] The risk management of such firms is to *increase* their corporate risk, although preferably in a very deliberate fashion.)

Hedging goods, exchange rates, interest rates, etc.

Because this is a chapter on options, we shall focus primarily on buying and selling contracts in the financial markets. The three most common risks that companies hedge are the prices of input or output goods (especially commodities), currency exchange prices, and interest rates. Hedging them is conceptually the same, so we can cover all of them together:

1. In the real world, the firm decides what it wants to hedge (e.g., its costs, sales, or income) and then determines its exposure to this risk.

   Some firms know their exposures from the operations of their actual businesses. For example, in 2005, Southwest Airlines spent about \$1.3 billion on jet fuel, about 20% of its operating expenses. Thus, it knew that a 5% rise in fuel prices would increase its operating expenses by \$65 million.

   Other businesses have to estimate their risks. For example, even a domestic U.S. firm may find that its U.S. customers tend to buy less of its product when the yen becomes cheaper. In this case, it must first determine its exposures. This is often done through a historical regression in which the firm's sales are explained by the underlying base asset (here, the exchange rate). For example, our firm may have run a regression of monthly sales on the exchange rate to find

   $$\text{Sales (in Millions)} \;=\; \$10 - \$0.05 \cdot (¥/\$) + \text{Noise} \tag{27.2}$$

   This suggests that if the current exchange rate is 100 ¥/\$, expected sales should be around $\$10 - \$0.05 \cdot (100) = \$5$ million. More importantly, it suggests that if the exchange rate increases to 101 ¥/\$ (that is, the yen becomes cheaper because you get more yen per dollar), sales would be expected to decline to \$4.95 million. Thus, this firm has a sales exposure of \$50,000 for each 1-yen change in value. This is exactly what the 0.05 regression coefficient gives you—it is your hedge ratio, the same as the delta in the Black-Scholes formula.

2. The firm decides how much of its risk it wants to hedge. Reducing risk has not only an upside but also a downside. For example, if an airline buys jet fuel today, it is a great hedge against future fuel price increases, but it will hurt its profitability if the fuel price decreases. An airline may also suffer other maladies and may not need as much fuel as it originally anticipated. And there is a cost to executing fuel

hedges. Then there are strategic considerations—if the airline is very different from its competitors, it may go out of business in the most likely scenario, but it could really pounce and gobble up its competitors if the less likely scenario occurs. Thus, hedging can create a real option!

A real-world example: Southwest's lucky fuel hedging program.

Firms do not need to disclose all their hedges. Indeed, hedging operations are often so complex and multifaceted that it may not even be possible to disclose them fully within the confines of a typical financial statement. Although we do not have full knowledge of how firms are hedging, we do have some data from certain industries. Research by Carter, Rogers, and Simkins shows that about two-thirds of U.S. airlines engaged in active hedging programs from 1992 to 2003. But during that time no airline hedged even 1 full year of jet fuel consumption. They typically hedged only about 15% of their annual fuel purchases. The two most active hedgers were Southwest and JetBlue, which hedged 43% of their annual fuel purchases. (By 2005, Southwest had significantly scaled up its fuel hedging operations—and to its good fortune. In 2005, it yielded a positive $892 million inflow vis-à-vis its $1.3 billion fuel cost.) Of course, even if an airline hedges its entire fuel budget for 1 year, if fuel prices rise, it would likely affect not only the next year but many years thereafter. This means that its lifetime operating costs would still remain quite exposed to fuel price risk. In this long-run sense, most corporate hedging programs seem conservative.

We have only scratched the tip of the iceberg—the devil is in the details.

This is only a small taste of risk management. In the real world, there are many other complications. For example, firms need to consider what exactly they should hedge—operating costs may not be the right target. After all, it could be that firms can charge customers higher prices when their input costs are higher. Higher input costs may not be detrimental—in fact, some financially strong firms may even benefit from otherwise adverse economic price developments if their competitors are forced out of business. Another hedging consideration is more technical: What firms want to hedge may not be linearly related to the underlying commodity, as it was in Formula 27.2. This can often be dealt with through dynamic trading (the same concept underlying the Black-Scholes formula). Yet another common problem is that the commodity available for hedging may not be the exact commodity that the firm wants to hedge. (It may only have short-term crude oil futures to trade, while it would really want to buy long-term jet fuel.) This can create all sorts of mismatching trouble. In any case, the firm may have to make some interim payments on its hedges and so has to worry about having enough liquidity before its own investments mature. This can also have certain accounting reporting obligations, which could in turn trigger certain bond covenants.

Q 27.19. Assume that oil is trading for $50 per barrel today. The oil price can go down by 33% or up by 50% per year. That is, it can sell for either $33.33 or $75.

1. You own a refinery. It is worth more if the oil price is higher. Intuitively, what kind of oil transaction would reduce your risk?
2. Your refinery can produce profits of $1.5 million if oil trades for $33.33, and profits of $3 million if it trades for $75. If you write a contract to sell 30,000 barrels of oil for $50/barrel next year, how would your risk exposure change?
3. If you want to be fully hedged, how many barrels of oil should you be selling?

ANECDOTE **223 years of Barings, 1 year of Leeson. (And Societe Generale and UBS and ...)**

Derivatives can be powerful hedging tools. But they can also be powerful speculation tools.

- In 1994, Barings was a venerable 223-year-old London investment bank. It had financed the Napoleonic Wars and the Louisiana Purchase. However, Barings was not equipped to handle its own 28-year-old trader Nick Leeson in its Singapore branch office. Leeson lost $1.3 billion—the entire equity assets of Barings—in a series of bets using options on forwards on the Nikkei index. (Like any other derivatives, these Nikkei options can be used either for hedging or for speculation.) One lesson from Leeson is that becoming notorious is not a bad way to earn large fees on the after-dinner speaking circuit. (He did however have to pay a 4 year sentence in a Singapore prison.)
- In 2008, Societe Generale disclosed that a 31-year-old rogue trader, Jerome Karviel, had lost $7 billion. If the gambles had paid off over the years, he would have probably been paid double-digit-million-dollar bonuses. Instead, having lost, he got 3 years in jail. (He is currently appealing this sentence. Still, even 50% times $100 million outweighs 50% times 3 years in jail in my book.) Unlike Barings, Societe Generale was large enough to survive this loss.
- In 2011, UBS disclosed that its own 31-year-old rogue trader, Kewku Adoboli, had lost $2.3 billion. UBS's press release stated that "The positions taken were within the normal business flow of a large global equity trading house as part of a properly hedged portfolio. However, the true magnitude of the risk exposure was distorted because the positions had been offset in our systems with fictitious, forward-settling, cash ETF positions, allegedly executed by the trader. These fictitious trades concealed the fact that the index futures trades violated UBS's risk limits." (Reader: the lesson is *not* to avoid 31-year old traders.)

Rogue "hedging" operations are common. For every rogue trader with losses large enough to require public disclosure (with public embarrassment for management), there are probably ten rogue traders whose losses are just in the hundred million dollar range—small enough not to have to be disclosed. The corporate lesson from these episodes is that inadequate oversight of financial traders—who would have earned large bonuses on their trading profits if their gambles had won—can easily make the risk of a firm worse, not better. *Firms need good risk management for their risk management.*

BBC and Other Sources

Q 27.20. Is it possible for a small firm to hedge the risk of overall stock market (S&P 500) movements? That is, could a firm with a market beta of 1.5 change its market beta to 0? If so, have you seen its hedge ratio (delta) before?

Employee Stock Options

Employee options differ from ordinary financial options.

Many firms have managerial and **employee stock option plans** (**ESOP**) in order to better motivate their workforce. The main idea is that options are more sensitive to changes in the underlying value of the firm than stock, so employees will be especially motivated to work hard if they own options. There are many unusual details to these employee options:

- They tend to be very long term (often as long as 10 years).
- They often vest only after several years (meaning that if the employee leaves the firm before that time, he loses the option).

- They are actually misnamed. If exercise triggers the creation of new underlying shares by the firm, then the proper name for such a claim would be a **warrant**, not an option. This is the case here: Almost all employee stock options are dilutive.
- Because of tax rules, most of these options must have a strike price equal to the current underlying stock price.
- Most importantly, they cannot be sold or bought, and because employees are often not allowed to short the firm's stock or own put options on it, these options cannot be easily hedged by employees. This should not be surprising—after all, the very reason the firm gives its employees these options is to leave them exposed to the fortunes of the firm.

➤ *Warrants and dilution, Pg.??.*

The last feature means that employee stock options are very different from other financial options. There is no hedge that forces their value. On the contrary—they are worth less to employees than they would be to third parties. To say it again: The firm gives its employees a security that costs more than what employees value it for. In the extreme, if employees are extremely risk averse, they may not place any value ex-ante on these options. Moreover, employees should exercise their options as soon as they can in order to diversify their wealth away from being too linked to this one company. From the perspective of the company, early exercise reduces the options' effective costs when compared with a hypothetical issue of freely trading warrants to external investors. But early exercise also robs the firm of the options' incentive effects sooner—which was, after all, the whole point of granting these options. Our tools, like the Black-Scholes formula or put-call parity, are definitely not applicable in this context.

Employee and executive option grants can be very big, especially, but not only, in high-tech firms.

Executive options are not small potatoes. For example, in April 2002, *Business Week* reported that Larry Ellison, CEO of Oracle, had pocketed $706 million from the exercise of long-held stock options—more than the GDP of Grenada! "Fortunately," Oracle stock was off 57% that year, or Ellison's options would have been worth $2 billion more. That same year, Dennis Kozlowski, CEO of Tyco, hit number 3 on the executive payoff list. However, he wound up in jail, partly for criminally looting $600 million from Tyco. (Maybe he should have received more options!)

ANECDOTE **2006 GAAP Change in the Treatment of Executive and Employee Options**

Executive options seemed particularly attractive to firms prior to 2006, because U.S. GAAP did not require firms to expense these options. Thus, these options did not have a negative influence on firms' financial statements upon granting—they were almost invisible as far as the firms' financials were concerned. (Of course, this was highly misleading. Even if not exercised, options can have tremendous value at issue time. They are *not* free to the corporation.) The adoption of this option-expensing rule by FASB in 2004 provoked strong complaints by many firms, especially high-tech firms. Even the U.S. Senate did some grandstanding with a motion to strike down this rule.

However, this storm of indignation died down in the wake of another scandal. Articles by David Yermack (from New York University) and others showed that many of these executive options were (illegally) backdated. That is, many corporate boards claimed to have granted options to their executives a number of days earlier when/if the stock price was lower in order to artificially increase the option value.

Summary

This chapter covered the following major points:

- Call options give the right (but not the obligation) to purchase underlying securities at a predetermined strike price for a given period of time. Put options give the right (but not the obligation) to sell underlying securities at a predetermined strike price for a given period of time. American call options give this right all the way up to the final expiration; European call options give this right only at the final expiration.
- Option payoffs at expiration and complex option strategies are best understood by graphing their payoff diagrams.
- A number of static no-arbitrage relationships limit the range of prices that an option can have.
- The most important no-arbitrage relationship is put-call parity, which relates the price of a call to the price of a put, the price of the underlying stock, and the interest rate.
- Put-call parity implies that American call options are never exercised early, and therefore that American calls are worth the same as European options. (This assumes no dividends.)
- The Black-Scholes formula relates the price of a call to five input parameters. The Black-Scholes value increases with the stock price, decreases with the strike price, increases with the time left to maturity, increases with the volatility, and increases with the risk-free interest rate.
- Options techniques and insights have found applications in the valuation of corporate securities, in capital budgeting of projects that allow for future flexibility (real options), and in risk management. They are less easy to apply in the context of employee and executive stock option plans.

Keywords

Answers

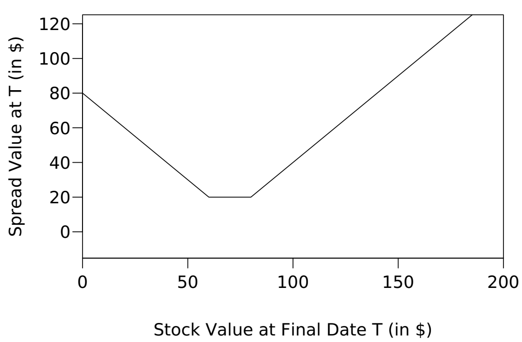
Q 27.1 Owning a call option is similar to selling a put option in that both are bullish bets. However, they have *very* different payoff patterns (tables). For example, the owner of a call option enjoys limited liability and thus can, at most, lose the money paid for the call. The seller of a put option can lose an unlimited amount.

Q 27.2 An option that is far in-the-money and expiring soon will change in value about one to one with the underlying stock price. After all, it will almost surely pay off.

Q 27.3 A put option holder is indifferent to the stock split in a perfect market because the contract is such that the option would be adjusted. However, the unexpected dividend increase would be good news for a put holder. In a perfect market, there would be no value change to the dividend announcement, but the post-dividend price at expiration would be lower.

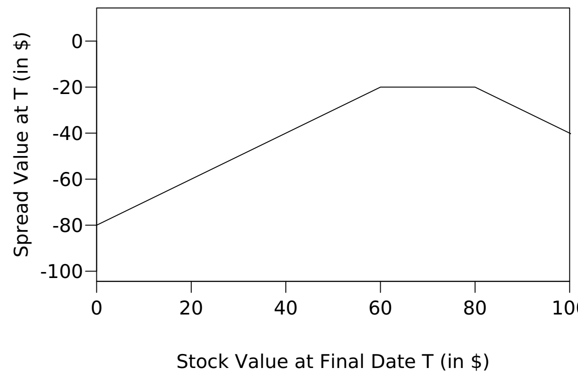
Q 27.4 The long call option with a strike price of \$60 pays off if the stock price ends above \$60; the long put option with a strike price of \$80 pays off if it ends up below \$80:

Stock	Pfio	Stock	Pfio
\$0	+\$80	\$70	+\$20
\$20	+\$60	\$75	+\$20
\$40	+\$40	\$80	+\$20
\$60	+\$20	\$90	+\$30
\$65	+\$20	\$100	+\$40



Q 27.5 The short call option with a strike price of \$60 costs money if the stock ends up above \$60; the short put option with a strike price of \$80 costs money below \$80:

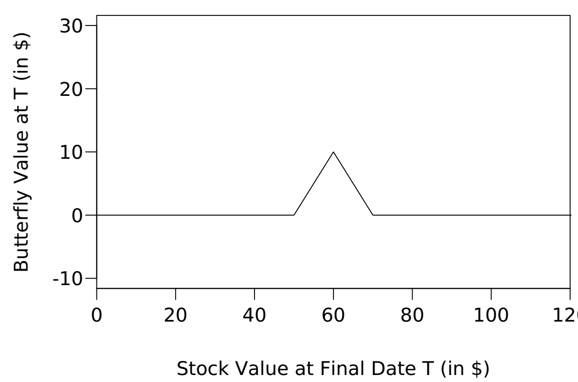
Stock	Pfio	Stock	Pfio
\$0	–\$80	\$70	–\$20
\$20	–\$60	\$75	–\$20
\$40	–\$40	\$80	–\$20
\$60	–\$20	\$90	–\$30
\$65	–\$20	\$100	–\$40



Q 27.6 The butterfly spread (1 long call K=\$50, 2 short calls K=\$55, 1 long call K=\$60):

Stock Now	1 Long Call K=\$50	2 Short Calls K=\$55	1 Long Call K=\$60	Net Pfio
⋮				
\$40	\$0	\$0	\$0	\$0
\$50	\$0	\$0	\$0	\$0
\$52	\$2	\$0	\$0	\$2
\$53	\$3	\$0	\$0	\$3
\$55	\$5	\$0	\$0	\$5
\$57	\$7	–\$4	\$0	\$3
\$58	\$8	–\$6	\$0	\$2
\$60	\$10	–\$10	\$0	\$0
\$65	\$15	–\$20	\$5	\$0
\$70	\$20	–\$30	\$10	\$0
⋮				

The payoff diagram of this butterfly spread is



Q 27.7 Put-call parity is the formula $C_0(K) = P_0(K) + S_0 - PV_0(K)$. The price of a call option now, the price of the same put option (strike price and expiration time) now, the stock price now, and the present value now of the strike price are the inputs.

Q 27.8 1. Put-call parity states that $C_0(K) = P_0(K) + S_0 - PV_0(K)$. Therefore, $P_0(K) = C_0(K) + PV_0(K) - S_0 = \$20 + \$80/1.10 - \$70 \approx \$22.73$.

2. The put option should cost \$22.73, but it indeed costs \$25.00. Therefore, it is too expensive, and you definitely need to short it. To cover yourself after shorting it, you now need to "manufacture" an artificial put option to neutralize your exposure. Put-call parity is $P_0(K) = C_0(K) + PV_0(K) - S_0 \approx \$22.73$. Loosely translated, a long put is a long call, a long present value of a strike price, and a short stock. Try purchasing one call (outflow now), saving the present value of the strike price (outflow), and shorting the stock (inflow now):

		S_T**<\$80**	
Execute	**Now**	**\$60**	**\$70**
+ 1 Call (K=\$80):	–\$20.00	\$0	\$0
- 1 Share:	+\$70.00	-\$60	-\$70
Save $PV_0(\$80)$:	–\$72.73	+\$80	+\$80
- 1 Put (K=\$80):	+\$25.00	–\$20	–\$10
Net	+\$2.27	\$0	\$0

	$S_T = \$80$	$S_T > \$80$	
Execute	**\$80**	**\$90**	**\$100**
+ 1 Call (K=\$80):	\$0	\$10	\$20
- 1 Share:	–\$80	–\$90	–\$100
Save $PV_0(\$80)$:	+\$80	+\$80	+\$80
- 1 Put (K=\$80):	\$0	\$0	\$0
Net	\$0	\$0	\$0

You would earn an immediate arbitrage profit of \$2.27.

Q 27.9 A European option can be worth as much as the equivalent American option if there is no value to early exercise. This happens if the option is a call option on a stock that pays no dividends.

Q 27.10 To compare the value of a live put to a dead put, compute the net value of a live put $(C_0(K) + [PV_0(K) - S_0])$ minus that of a dead put $\{(K - S_0)\}$. It is $\{C_0(K) + [PV_0(K) - S_0]\} - \{(K - S_0)\}$. This can be simplified into $C_0(K) + [PV_0(K) - K]$. This expression is worth more if the call is worth more (the stock price is high relative to the strike price) and if the interest rate is low. It is under those circumstances that you should not exercise the American put because it is worth less dead than alive. (In the real world, many put options that are far out-of-the-money have already been purchased *and* exercised before the final date, so they are no longer available.)

Q 27.11 Think about what a call with infinite time to maturity and strike price of \$0 really is—it is simply the stock itself. The (Black-Scholes) answer is that this must be equivalent to owning the underlying stock itself. Therefore, $C_0 = S_0 = \$80.50$.

Q 27.12 The present value of \$75 is $PV(\$75) = \$75/(1.05^{1/4}) \approx \$74.09$. Thus,

$$d_1 \approx \frac{\log_N(\$80/\$74.09)}{20\% \cdot \sqrt{0.25}} + \frac{1}{2} \cdot 20\% \cdot \sqrt{0.25} \approx 0.817,$$

so $\mathcal{N}(0.817) \approx 0.793$. Next, compute $d_2 = 0.817 - 20\% \cdot \sqrt{0.25} \approx 0.717$ and $\mathcal{N}(0.717) \approx 0.763$. Therefore, $BS(S_0 = \$80, K = \$75, T = 1/4, r_F = 5\%, \sigma = 20\%) = \$80 \cdot 0.793 - \$74.09 \cdot 0.763 \approx \$6.89$.

Q 27.13 To price the IBM put option:

1. First compute the European Black-Scholes call value: $BS(S = \$80.50, K = \$100, r_F = 1.77\%, t = 0.1333, \sigma = 30\%)$. The interest rate to maturity is $1.0177^{0.1333} \approx 1.00234$. Thus, the present value of the strike price is $PV(\$100) \approx \$100/1.00234 \approx \$99.767$. Next,

$$d_1 \approx \frac{\log_N(\$80.50/\$99.767)}{30\% \cdot \sqrt{0.1333}} + \frac{1}{2} \cdot 30\% \cdot \sqrt{0.1333}$$
$$\approx -1.9589 + 0.05477 \approx -1.904$$

and $\mathcal{N}(d_1) \approx 0.02845$. Then $d_2 \approx -2.0136$ and $\mathcal{N}(d_2) \approx 0.02202$. The call price is therefore about $BS(\$80.50, \$100, 0.1333, 1.77\%, 30\%) \approx \$80.50 \cdot 0.02845 - \$99.767 \cdot 0.02202 \approx \$2.289 - \$2.196 \approx \$0.0928$. Therefore, the European IBM put would be worth $\$0.0928 - \$80.50 + \$99.767 \approx \$19.36$. (Your answer may vary a little due to rounding.)

2. If you hold onto the put if it is American, you have an asset worth \$19.36. If you exercise it, you receive an immediate \$100-\$80.50=\$19.50. Therefore, you would be better off exercising immediately!

Q 27.14 The delta of an option is the number of stocks that you need to purchase in order to mimic the option. Delta is also called the hedge ratio.

Q 27.15 The value of a call option increases with higher share prices, longer lengths to maturity, more volatility, and higher interest rates; it decreases with higher strike prices.

Q 27.16 Not only is it possible to have a security that is an option on an option, but the fact is that almost all common financial options are such. This is because the stock on which they are written is itself an option on the underlying firm value. Thus, CBOE options are essentially options on options.

Q 27.17 Let's price the lease in 18 months. Assume that you must decide to produce at the start of this week. If you see that the price of gold is above \$400, then you extract gold. Otherwise, you do not. You can now value the gold mine as if it were 100 Black-Scholes call options, each with current price \$365, strike price of \$400, interest rate of 10%, volatility of 40% per annum, and 18 months to expiration. You can calculate this. The present value of the strike price is $PV(K) = \$400/1.1^{1.5} \approx \$346.71$. $\log_N[S/PV(\,)K)] \approx \log_N(\$365/\$346.71) \approx 0.0514$. The Black-Scholes value of such a call is about $BS(S = \$365, K = \$400, t = 1.5, r = 0.1, \sigma = 0.4) \approx \$79.51$. Thus, 100 calls should be worth \$7,951.

Q 27.18 The value of the mine would be the sum of many such options. The production cost per ounce increases by about $20\%/52 \approx 0.35\%$ per week. It would increase the strike price from \$400 to \$401.40, then to \$402.81, and so on.

$$\text{Value} = \text{BS}(S = \$365, K = \$400, t = 1, r = 10\%, \sigma = 40\%(\ ))$$
$$+ \text{BS}(S = \$365, K = \$401.40, t = 2, r = 10\%, \sigma = 40\%)$$
$$+ \text{BS}(S = \$365, K = \$402.81, t = 3, r = 10\%, \sigma = 40\%) + \cdots$$

Q 27.19 Given this process on the price of oil:

1. Selling oil would reduce your risk.
2. If you have agreed to sell 30,000 barrels of oil for \$50/barrel, you would receive \$1.5 million. If the oil price were to be \$33.33/barrel, you can buy 30,000 barrels for \$1 million. This would give you a net profit of \$0.5 million. If the oil price were to be \$75/barrel, you can buy the barrels for \$2.25 million. This would give you a net loss of \$0.75 million. Putting this together with your refinery, your payoffs would now be \$1.5+\$0.5=\$2 million if oil goes down, and \$3-\$0.75=\$2.25 million if oil goes up. Your risk is much lower now.
3. If you contract on 36,000 barrels of oil, your net is \$2.1 million in either case:
   - If oil drops to \$33.33, the gain on your hedge is $(\$50 - \$33.33) \cdot 36,000 = \$600,120$. Thus, your payoffs would be $\$1.5 + \$0.6 \approx \$2.1$ million.
   - If oil rises to \$75.00, the loss on your hedge is $(\$75 - \$50) \cdot 36,000 = \$900,000$. Thus, your payoffs would be $\$3 - \$0.9 \approx \$2.1$ million.

The 36,000 (x=36) was obtained by solving $\$1,500 + (\$50 - \$33.33) \cdot x = \$3,000 - (\$75 - \$50) \cdot x$.

Q 27.20 A firm could easily hedge its S&P 500 risk by shorting the stock market. This is cheaply done by trading S&P 500 futures or forwards. If the firm is worth \$100 million and has a beta of 1.5, shorting \$150 million in this future should do the trick. The hedge ratio is really the market beta itself!

End of Chapter Problems

Q 27.21. Is writing a call the same as buying a put, provided both have the same strike price and same expiration date? That is, do they give the same payoffs in future states of the world?

Q 27.22. An option is far out-of-the-money and will expire tonight. How would you expect its value to change when the stock price changes?

Q 27.23. Would a call option writer welcome an unexpected stock split? Would a call option writer welcome an unexpected dividend increase? (Assume a perfect market in both scenarios.)

Q 27.24. Write down the payoff table and draw the payoff diagram (both at expiration) of a portfolio consisting of one *short* call with a strike price $K = \$60$ and one *long* put with a strike price K= \$80.

Q 27.25. Graph the payoff diagram for the following straddle: one long call option with a strike price of \$50 and one long put option with a strike price of \$60.

Q 27.26. How could you earn money in the put-call parity example in Section 27.B if the 1-year put option traded in the market for \$25 per share, the stock price were \$80, the equivalent 1-year call cost \$30, and the interest rate were 10% per year?

Q 27.27. A 1-year put option with a strike price of \$80 costs \$25. A share costs \$70. The interest rate is 8% per year. What should a 1-year call option with a strike price of \$80 trade for?

Q 27.28. List and describe the simple no-arbitrage relationships, preferably both in words and in algebra.

Q 27.29. How would you cook up a numerical example in which you would want to exercise an American put before expiration? Is your American put in-the-money or out-of-the-money?

Q 27.30. What is the value of a call option with a strike price of \$0 and 6 months to expiration? Use the parameters of the example: S_0=\$80.50, r_F =1.77%, and $\sigma = 50\%$.

Q 27.31. Write a computer spreadsheet that computes the Black-Scholes value on row 4 as a function of its five inputs (in the first two rows). This will teach you more about the Black-Scholes formula than all the pages in this book. Recall that the normal distribution function is **normsdist**.

Q 27.32. Use your spreadsheet from Question 27.31 to price a call option with a stock price of $80, a strike price of $75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock. Check it against the solution in Question 27.12.

Q 27.33. Price the earlier call option but with a higher *strike price.* That is, price a call with a stock price of $80, a strike price of $80, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.

Q 27.34. Price the earlier call option with a higher *interest rate.* That is, price a call with a stock price of $80, a strike price of $75, 3 months to maturity, a 10% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.

Q 27.35. Price the earlier call option with a higher *volatility.* That is, price a call with a stock price of $80, a strike price of $75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 30% on the underlying stock.

Q 27.36. Price a *European put option* with a stock price of $80, a strike price of $75, 3 months to maturity, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.

Q 27.37. Price a *European straddle*: one call and one put option on a stock with a price of $80, both with strike prices of $75, a 5% risk-free rate of return, and a standard deviation of return of 20% on the underlying stock.

1. What is the price of the position if there are 3 months to maturity?
2. What is the price if nothing changed and there is only 1 month left to maturity?
3. What is the price at expiration?

Q 27.38. (Advanced) There are numerous calculators on the Web that will calculate an implied volatility for you. Fortunately, it is not difficult to write one yourself in a computer spreadsheet, using the built-in equation solver. Write a computer spreadsheet program that uses this equation solver to back out a volatility estimate, given a call price and the five Black-Scholes inputs. Use it to confirm the implied volatilities in Exhibit 27.5. Then use your spreadsheet and data from a financial website to compute the implied volatility of IBM now. Be clear about what inputs you are using.

Q 27.39. Are the deltas of options with different strike prices different?

Q 27.40. Using the computer spreadsheet you created in Question 27.31, graph the Black-Scholes value as a function of the current stock value for options with two different interest rates: 5% and 20%. That is, repeat Exhibit 27.7 for a 3-month option with strike price K=$90, 3 months to expiration, and a 20% volatility.

Q 27.41. Using the computer spreadsheet you created in Question 27.31, graph the Black-Scholes value as a function of the current stock value for options with three different volatilities: 20%, 80%, and 160%. That is, repeat Exhibit 27.7 for a 3-month option with strike price K=$90, 3 months to expiration, and a 5% interest rate.

Q 27.42. In words, how does the value of a call option change with the Black-Scholes inputs?

Q 27.43. Should employees and firms value employee stock options using the Black-Scholes formula?

End of Chapter Problems

Q 27.44. Price an American call option with a strike price of $53 over the last two instants before expiration.

Q 27.45. Price a European put option with a strike price of $53 over the last two instants before expiration. How does its value differ from an American Put option? (Hint: for the American put, consider at each node whether you would want to exercise the put or continue to hold it.)

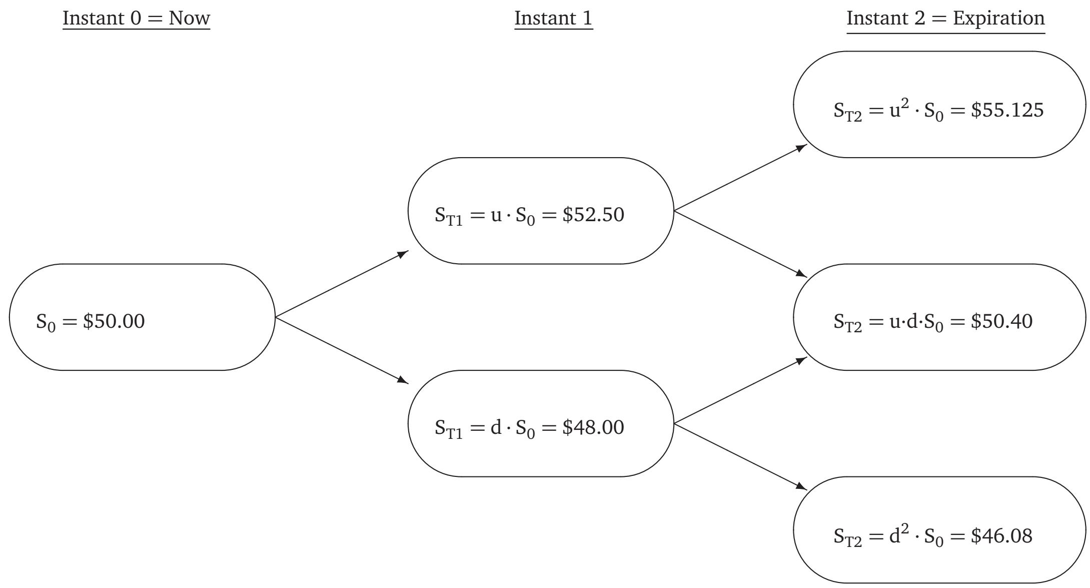
27.F Chapter Appendix: The Ideas behind the Black-Scholes Formula

In the previous sections, you learned how to use the Black-Scholes formula. However, it descended on you out of the ether. If you are wondering where the formula actually comes from, then this section is for you.

Modeling the Stock Price Process as a Binomial Tree

Assume at each instant, the stock prices go up by *u* or down by *d*.

The basic building element for the Black-Scholes formula is the assumption that over one instant, the stock price can only move up or down. (This is called a **binomial process**.) So you must first understand how to work in such a world. Over two instants, the stock price can move up twice, move up once and move down once, or move down twice. Use the letter u to describe the stock price multiplier when an up move occurs, and d to describe the stock price multiplier when a down move occurs. You can represent the stock price process with a binomial tree—where one branch represents a price-up movement and the other a price-down movement. For example, if d= 0.96 (which means that on a down move, the stock price declines by 4%) and u= 1.05 (the stock price increases by 5%), the stock price is as follows:



There are more outcomes (i.e., higher probability of reaching the nodes) in the middle of a binomial tree than at the edges.

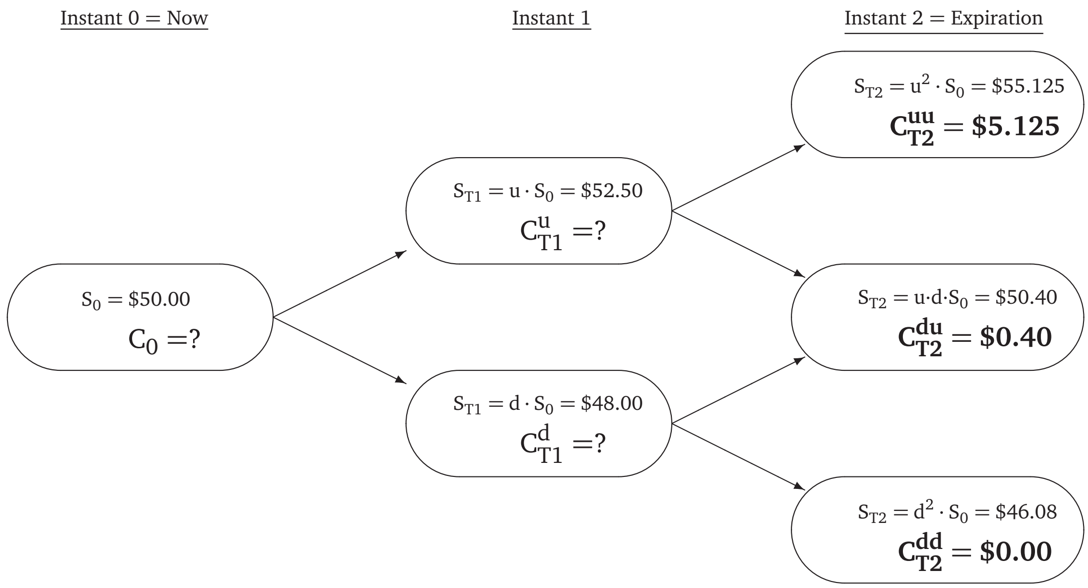
Note that at instant 2, the middle outcome occurs on two possible paths, while the two extreme outcomes occur only on one path each; $u \cdot d \cdot S_0$ can come about if there is one u followed by one d, or if there is one d followed by one u. This is already a statistical distribution that shares with a bell-shaped (normal) distribution the feature that middle outcomes are more likely than extreme outcomes. (With many more binomial tree levels, you indeed end up with a continuous distribution that looks a lot like a bell-shaped curve.)

➤ *Bell shaped distributions and binomial trees, Sect.* 27.F, *Pg.358*.

The Option Hedge

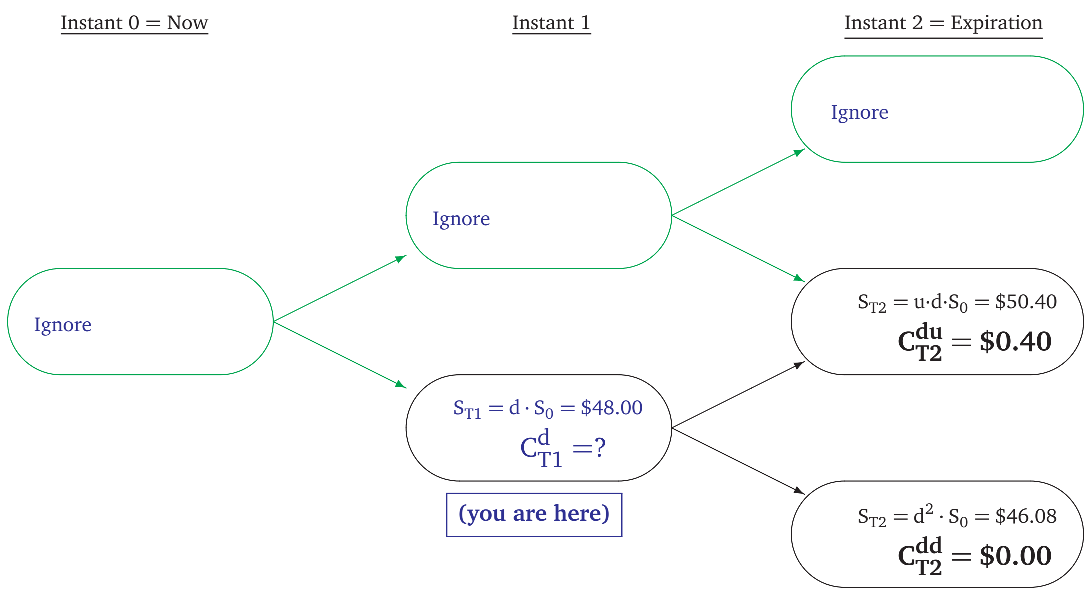
First, set out your goal: Find the value of an option two instants before expiration.

If you know that your stock follows this binomial process, and you know u and d, can you price a call option with a strike price of \$50? On inspection of the tree, realize that the call option pays \$0 if the stock price moves down twice, \$0.40 if the stock price moves up once and down once (or vice versa), and \$5.125 if the stock price moves up twice.



Price the call in the down state (C^d_1).

Your ultimate goal is to determine the call price at the outset, C_0. First place yourself into the position where the stock price has moved down once already, that is, where the stock price stands at $48.00.



Simplifying the pricing with formulas.

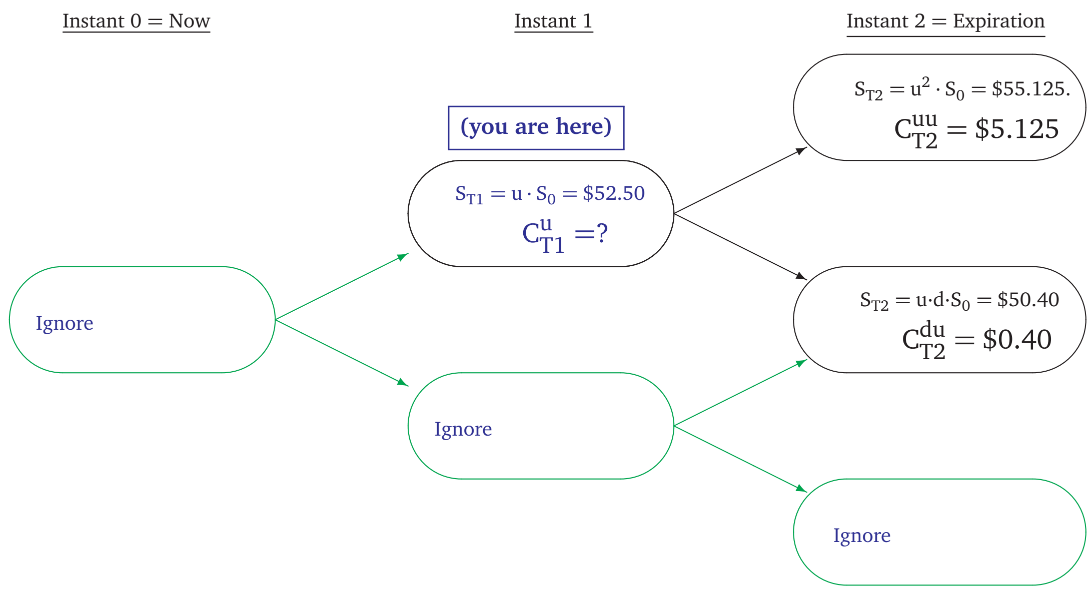
Your immediate goal is to buy stocks and risk-free bonds so that you receive $0 if the stock moves down and $0.40 if the stock moves up. Assume you purchase δ stocks and b bonds. Bonds increase at a risk-free rate of 1+0.1% each instant. If you own δ stock and the stock price goes up, you will own $\delta \cdot u \cdot S_0$ stock. If you own δ stock and the stock price goes down, you will own $\delta \cdot d \cdot S_0$stock. Can you purchase a particular δ amount of stock and a particular b amount of bonds to earn exactly the same as your call option? Solve for b and δ so that

$$\begin{aligned}
\delta \cdot 0.96 \cdot \$48 + b \cdot (1.001) &= \$0.00 \\
\delta \cdot 1.05 \cdot \$48 + b \cdot (1.001) &= \$0.40 \\
\delta \cdot d \cdot S_0 + b \cdot (1+r) &= C_d \\
\delta \cdot u \cdot S_0 + b \cdot (1+r) &= C_u
\end{aligned}$$

The solution is

$$\delta = \frac{\$0.40 - \$0.00}{1.05 \cdot \$48 - 0.96 \cdot \$48} \approx 0.0926 \quad \text{and} \quad b \approx -\$4.262$$

If you purchase a portfolio of 0.0926 shares (which costs $0.0926 \cdot \$48 \approx \$4.444$) and borrow \$4.262 (for a net outlay of \$0.182 now), then in the next period, this portfolio will pay off \$0 in the downstate and \$0.40 in the upstate. Because this is exactly the same as the payoff on the call option, the C_1^d call option should also be worth \$0.182. This is the law of one price (absence of arbitrage) in action.



Repeat the option pricing in the up state (C_1^u).

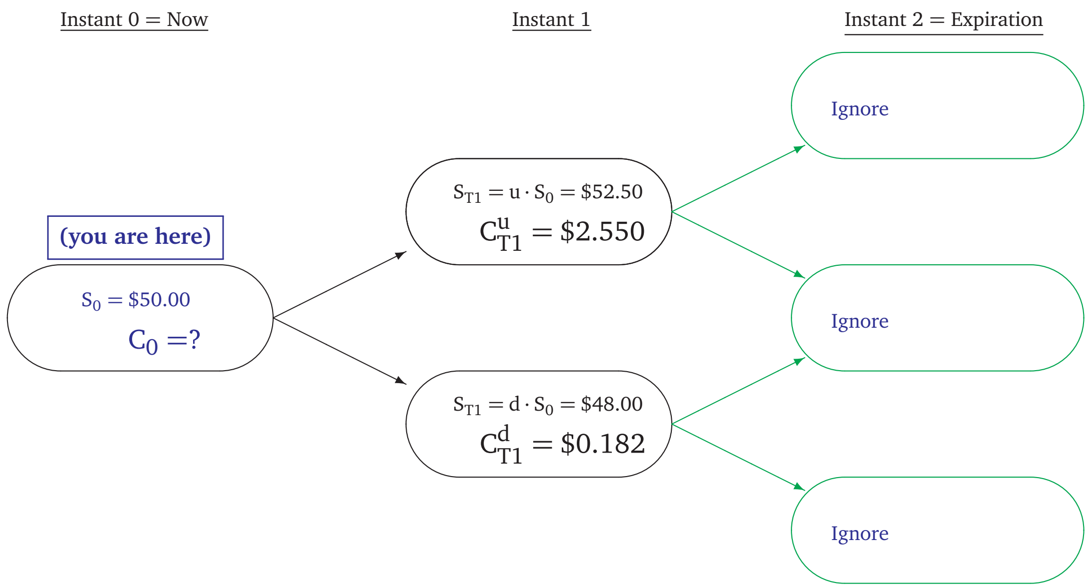
Now repeat the same exercise where the stock price stands at \$52.50 and next instant you can end up with either \$0.40 in the downstate or \$5.125 in the upstate. In this case, solve

$$\begin{aligned}
\delta \cdot 0.96 \cdot \$52.50 + b \cdot (1.001) &= \$0.400 \\
\delta \cdot 1.05 \cdot \$52.50 + b \cdot (1.001) &= \$5.125 \\
\delta \cdot d \cdot S_0 + b \cdot (1+r) &= C_d \\
\delta \cdot u \cdot S_0 + b \cdot (1+r) &= C_u
\end{aligned}$$

And the solutions are

$$\delta = \frac{\$5.125 - \$0.40}{1.05 \cdot \$52.50 - 0.96 \cdot \$52.50} = 1.00 \quad \text{and} \quad b \approx -\$49.95$$

If you purchase 1.00 shares (at a price of \$52.50) and borrow \$49.95 (for a net portfolio cost of \$2.550), you will receive \$5.125 if the stock price goes up and \$0.40 if the stock price goes down. Therefore, after the stock price has gone up once to stand at \$52.50, the C_1^u call option has to be valued at \$2.550, too.



To determine the value of the call C_0 at the outset, find the price of a security that will be worth \$0.182 if the stock moves from \$50 to \$48, and worth \$2.55 if the stock moves from \$50 to \$52.50:

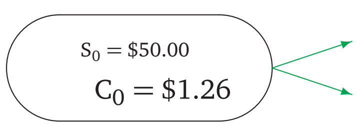
Repeat the option pricing now (C_0).

$$\begin{aligned}
\delta \cdot 0.96 \cdot \$50.00 + b \cdot (1.001) &= \$0.182 \\
\delta \cdot 1.05 \cdot \$50.00 + b \cdot (1.001) &= \$2.550 \\
\delta \cdot d \cdot S_0 + b \cdot (1 + r) &= C_d \\
\delta \cdot u \cdot S_0 + b \cdot (1 + r) &= C_u
\end{aligned}$$

The solution is

$$\delta = \frac{\$2.550 - \$0.182}{1.05 \cdot \$50 - 0.96 \cdot \$50} \approx 0.5262 \quad \text{and} \quad b \approx -\$25.05$$

You have to purchase 0.5262 shares (cost now: \$26.31), and borrow \$25.05 dollars. Your portfolio's total net outlay is $\$26.31 - \$25.05 \approx \$1.26$. Therefore, it follows that, by arbitrage, the price of the call option C_0 must be about \$1.26 now.



Matching a Stock Price Distribution to a Binomial Tree and Infinite-Level Pricing

In real life, the stock price can move many more times than just twice. You need a tree with many more levels, so you need to generalize this binomial process to more levels. For example, if there are 10 instants, what would be the worst possible outcome? Ten instant down movements mean that the stock price would be

$$\text{Worst-Case Scenario:} \quad d^{10} \cdot S_0 = 0.96^{10} \cdot \$50 \approx \$33.24$$

The second-worst outcome would be one instant of up movement, and nine instants of down movement.

$$\text{Second-Worst-Case Scenario:} \quad d^{9} \cdot u^{1} \cdot S_0 = 0.96^{9} \cdot 1.05^{1} \cdot \$50 \approx \$36.36$$

Although the worst scenario can only occur if there are exactly 10 down movements, there are 10 different ways to fall into the second-worst scenario, ranging from duuuuuuuuu, uduuuuuuuu, ..., to uuuuuuuuud. This should bring back bad memories of "combinations" from your SAT test: These are the 10 possible combinations, better written as

$$\binom{10}{1} = \frac{10!}{1! \cdot 9!} = 10$$

$$\binom{N}{i} = \frac{N!}{N! \cdot (N-i)!}$$

Therefore, with N levels in the tree, the stock price will be $u^{i} \cdot d^{N-i} \cdot S_0$ in $\binom{N}{i}$ paths.

The probability of exactly 1 in 10 up movements, if the probability of each up movement is 40%, would be

$$\text{Prob}(1\,u's, 9\,d's) = \binom{10}{1} \cdot 0.4^{1} \cdot (1-0.4)^{10} - 1 \approx 4\%$$

$$\text{Prob}((i)\,u's, (N-i)\,d's) = \binom{N}{i} \cdot p^{i} \cdot (1-p)^{N-i}$$

With many nodes, a binomial tree becomes a log-normal distribution.

Still, is it enough to work with such an unrealistic binomial tree process, given that the stock price from now to expiration is more likely to have a continuous bell-shaped distribution? Put differently, how realistic is this binomial stock price process? Exhibit 27.8 plots a distribution of prices at the end of the tree if there are up to 500 nodes, if up and downs are equally likely, and if u=1.02 and d = 1/u ≈ 0.98. This binomial process looks as if it can generate a pretty reasonable distribution of possible future stock price outcomes.

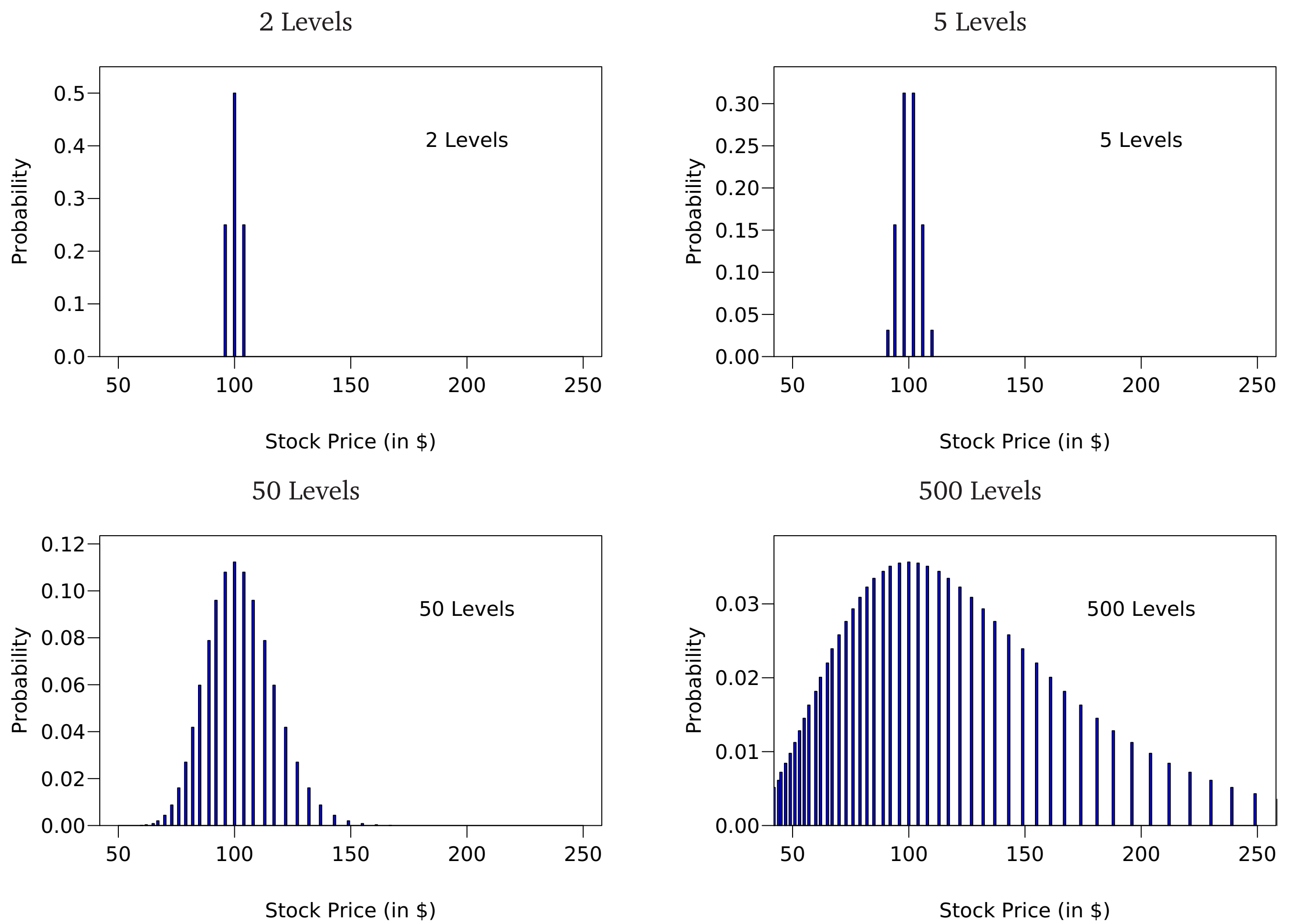


Exhibit 27.8: *Stock Price Processes Simulated via Binomial Processes.* The probability of an up movement at each tree node is 50-50. The value multiplier is u=1.02 if an up movement occurs, d=1/1.02 if a down movement occurs. The stock price is $100. The graphs differ in the number of levels in the tree: 2, 5, 50, and 500.

How the distribution works in the limit with infinitely many levels.

If you assume that the stock prices can only move up or down each instant and that there are an infinite number of instants, then the underlying stock price distribution follows a **log-normal distribution**, with $0 as the lowest possible outcome. The rate of return follows a log-normal distribution with –100% as the lowest possible outcome. (The log-normal name comes from the fact that if a variable P follows a log-normal distribution, then log(P) follows a normal distribution.)

A practical question is how to select u, d, and q (where q is the true probability of an up movement) in a simulated tree to match an empirically observed stock price distribution. Assume you have a historical rate of return series to provide you with a reasonable mean and a reasonable variance for the expected rate of return. Call *dt* a really tiny time interval, call m the mean that you want to match, and s the standard deviation. Then select u and d as follows:

$$\mathrm{u} = \mathrm{m} \cdot \mathrm{dt} + \mathrm{s} \cdot \sqrt{\mathrm{dt}} \quad \text{and} \quad \mathrm{d} = \mathrm{m} \cdot \mathrm{dt} - \mathrm{s} \cdot \sqrt{\mathrm{dt}}$$

In the limit, these choices create a log-normal distribution, which is completely characterized by its mean and variance, with mean m and standard deviation s.

Binomial Pricing and the Black-Scholes Formula

B-S is a tree with infinitely many of these nodes.

In sum, the process to price options is as follows:

1. Determine the real-world stock price distribution to expiration—most importantly, the stock volatility.
2. Compute the u and d that you need in order to build your tree with a great many levels to expiration—the more the better—to match the real-world stock price distribution.
3. After you have written down your tree, write down the payoff of your option as a function of the underlying stock on the final nodes.
4. Work your way backward through the binomial tree.
5. At the origin node, you can read off the amount of stock (delta) that you need to purchase in order to mimic your option. You can buy the underlying stock and borrow some funds so as to mimic exactly how your option can change in value over the next instant, and your net cost determines the value of the option.

Computers can do this extremely quickly. You can also use this technique to price options that you could not otherwise price. For example, to price an American put option, work your way backward through the tree, asking yourself at each node whether exercising your put option would yield greater profits than keeping it. If it would, assume you would exercise at this node, and use this higher value while working backward thereafter.

To find the Black-Scholes formula, there are no more novel concepts or intuition. You only need a lot of (tedious) algebraic manipulation and simplification. You let the number of levels in the tree go to infinity—of course, adjusting u and d in a way that continues to match the real-world stock volatility from now to expiration. After this messy algebra, the Black-Scholes formula pops right out. The amount of stock you need to purchase for your mimicking portfolio, which is δ in our binomial notation, becomes $\mathcal{N}(\mathrm{d}_1)$ in this limit. Done.

End of Chapter Problems

Q 27.46. Price an American call option with a strike price of \$53 over the last two instants before expiration.

Q 27.47. Price a European put option with a strike price of \$53 over the last two instants before expiration. How does its value differ from an American Put option? (Hint: for the American put, consider at each node whether you would want to exercise the put or continue to hold it.)