5

Time-Varying Rates of Return and the Yield Curve

When Rates of Return are Different

In this chapter, we will make the world a little more complex and a lot more realistic, although we are still assuming perfect foresight and perfect markets. The first assumption that we will abandon is that rates of return are the same no matter what the investment time horizon is. In the previous chapters, the interest rate was the same every period—if a 30-year bond offered an interest rate of 5% per annum, so did a 1-year bond. But this is not the case in the real world. Rates of return usually vary with the length of time an investment requires.

The U.S. Treasury Department Resource Center informs you of this fact every day. For example, on Dec 31, 2015, a U.S. Treasury paid 0.65% *per annum* for a payment to be delivered in one year (Dec 31, 2016), but 3.01% *per annum* for a payment to be delivered in 30 years (Dec 31, 2045):

	In Percent Per Annum										
	1m	3m	6m	1y	2y	3y	5y	7y	10y	20y	30y
12/31/2015	0.14	0.16	0.49	0.65	1.06	1.31	1.76	2.09	2.27	2.67	3.01

Is this stuff that just bond traders need to know? Not at all. In fact, this stuff matters to you, too. Have you ever wondered why your bank's one-month CD offered only 0.21% *per annum*, while their five-year CD offered 0.8% *per annum*? (These were the average national rates on Dec 31, 2015.) Which should you choose? Why? And does inflation matter?

CEOs must also know how to compare what they should have to pay for investors willing to give them money for a return promise in a 1-year project vs. a 30-year project. If the U.S. Treasury has to offer higher rates for lengthier investment return periods, surely so will firms!

In this chapter, you will learn how to work with time-dependent rates of return and inflation. In addition, this chapter contains an (optional) section that explains the U.S. Treasury yield curve.

5.1 Working with Time-Varying Rates of Return

Interest rates can differ based on the length of the commitment.

In the real world, rates of return usually differ depending on when the payments are made. For example, the interest rate next year could be higher or lower than it is this year. Moreover, it is often the case that long-term bonds offer different interest rates than short-term bonds. You must be able to work in such an environment, so let me give you the tools.

Compounding Different Rates of Return

A compounding example with time-dependent rates of return.

Fortunately, when working with time-varying interest rates, all the tools you have learned in previous chapters remain applicable (as promised). In particular, compounding still works exactly the same way. For example, what is the two-year holding rate of return if the rate of return is 20% in the first year and 30% in the second year? (The latter is sometimes called the **reinvestment rate**.) You can determine the two-year holding rate of return from the two 1-year rates of return using the same compounding formula as before:

$$(1 + r_{0,2}) = (1 + 20\%) \cdot (1 + 30\%) = (1 + 56\%)$$
$$(1 + r_{0,1}) \cdot (1 + r_{1,2}) = (1 + r_{0,2})$$

Subtract 1, and the answer is a total two-year holding rate of return of 56%. If you prefer it shorter,

$$r_{0,2} = 1.20 \cdot 1.30 - 1 = 1.56 - 1 = 56\%$$

The calculation is not conceptually more difficult, but the notation is. You have to subscript not just the interest rates that begin now, but also the interest rates that begin in the future. Therefore, most of the examples in this chapter must use two subscripts: one for the time when the money is deposited, and one for the time when the money is returned. Thus, $r_{1,2}$ describes an interest rate from time 1 to time 2. Aside from this extra notation, the compounding formula is still the very same multiplicative "one-plus formula" for each interest rate (subtracting 1 at the end).

The general formula for compounding over many periods.

You can also compound to determine holding rates of return in the future. For example, if the 1-year rate of return is 30% from year 1 to year 2, 40% from year 2 to year 3, and 50% from year 3 to year 4, then what is your holding rate of return for investing beginning next year for three years? It is

$$\text{Given:} \quad r_{1,2} = 30\% \quad r_{2,3} = 40\% \quad r_{3,4} = 50\%$$
$$(1 + r_{1,4}) = (1 + 30\%) \cdot (1 + 40\%) \cdot (1 + 50\%) = (1 + 173\%)$$
$$(1 + r_{1,2}) \cdot (1 + r_{2,3}) \cdot (1 + r_{3,4}) = (1 + r_{1,4})$$

Subtracting 1, you see that the three-year holding rate of return for an investment that takes money *next* year (not today!) and returns money in 4 years (appropriately called $r_{1,4}$) is 173%. Let's be clear about the timing. For example, say it was midnight of December 31, 2016, right now. This would be time 0. Time 1 would be midnight December 31, 2017, and this is when you would invest your \$1. Three years later, on midnight December 31, 2020 (time 4), you would receive your original dollar plus an additional \$1.73, for a total return of \$2.73. Interest rates that begin right now—where the first subscript would be 0—are usually called **spot rates**. Interest rates that begin in the future are usually called **forward rates**.

Q 5.1. If the first-year interest rate is 2% and the second year interest is 3%, what is the two-year total interest rate?

Q 5.2. Although a two-year project had returned 22% in its first year, overall it lost half of its value. What was the project's rate of return after the first year?

Q 5.3. From the closing of December 31, 2009 to December 31, 2015, Vanguard's S&P 500 fund (which received and paid dividends on the underlying constituent stocks to its fund investors, but charged administration fees) returned the following annual rates of return:

2010	2011	2012	2013	2014	2015
15.0%	2.1%	16.0%	32.3%	13.7%	1.3%

What was the rate of return over the first 3 years, and what was it over the second 3 years? What was the rate of return over the whole 6 years? Was the *realized* rate of return time-varying?

Q 5.4. A project lost one-third of its value the first year, then gained fifty percent of its value, then lost two-thirds of its value, and finally doubled in value. What was the average rate of return? What was the investment's overall four-year rate of return? If one is positive, is the other, too?

Annualized Rates of Return

Per-unit standard measures are statistics that are conceptual aids.

Time-varying rates of return create a new complication that is best explained by an analogy. Is a car that travels 163,680 yards in 93 minutes fast or slow? It is not easy to say, because you are used to thinking in "miles per sixty minutes," not in "yards per ninety-three minutes." It makes sense to translate speeds into miles per hour for the purpose of comparing them. You can even do this for sprinters, who run for only 10 seconds. Speeds are just a standard measure of the rate of accumulation of distance per unit of time.

A per-unit standard for rates of return: annualized rates.

The same issue applies to rates of return: A rate of return of 58.6% over 8.32 years is not as easy to compare to other rates of return as a rate of return per year. Therefore, most rates of return are quoted as **annualized rates**. The average annualized rate of return is just a convenient unit of measurement for the rate at which money accumulates—a "sort-of-average" measure of performance. Of course, when you compute such an annualized rate of return, you do not mean that the investment earned the same annualized rate of return of, say, 5.7% each year—just as the car need not have traveled at 60 mph (163,680 yards in 93 minutes) each instant.

Return to our example: You want to annualize our three-year total holding rate of return.

If you were earning a total three-year holding rate of return of 173% over the three-year period, what would your *annualized* rate of return be? The answer is not the **average rate of return** of 173%/3 ≈ 57.7%, because if you earned 57.7% per year, you would have ended up with $1.577^3 - 1 \approx 292\%$, not 173%. This incorrect answer of 57.7% ignores the *compounded interest on the interest* that you would earn after the first and second years. Instead, to compute the annualized rate of return, you need to find a single hypothetical annual rate of return that, if you received it each and every year, would give you a three-year holding rate of return of 173%.

To find the t-year annualized interest rate, take the t-th root of the total return (t is number of years).

How can you compute this? Call this hypothetical annual rate that you would have to earn each year for three years $r_{\overline{3}}$ (note the bar above the 3 to denote *annualized*) in order to end up with a holding rate of return of 173%. To find $r_{\overline{3}}$, solve the equation

$$(1 + r_{\overline{3}}) \cdot (1 + r_{\overline{3}}) \cdot (1 + r_{\overline{3}}) = (1 + 173\%)$$

$$(1 + r_{\overline{3}}) \cdot (1 + r_{\overline{3}}) \cdot (1 + r_{\overline{3}}) = (1 + r_{0,3})$$

or, for short,

$$(1 + r_{\overline{3}})^3 \;=\; (1 + 173\%)$$
$$(1 + r_{\overline{t}})^t \;=\; (1 + r_{0,t}) \qquad (5.1)$$

In our example, the holding rate of return $r_{0,3}$ is known (173%) and the annualized rate of return $r_{\overline{3}}$ is unknown. Earning the same rate ($r_{\overline{3}}$) three years in a row should result in a holding rate of return of 173%. It is a "smoothed-out" rate of return of the three years' rates of return. Think of it as a hypothetical, single, constant-speed rate at which your money would have ended up as quickly at 173% as it did with the 30%, 40%, and 50% individual annual rates of return. The correct solution for $r_{\overline{3}}$ is obtained by computing the third root of 1 plus the total holding rate of return:

➤ Math Background, Chapter A, Pg.631.

$$(1 + r_{\overline{3}}) \;=\; (1 + 173\%)^{(1/3)} \;=\; \sqrt[3]{1 + 173\%} \;\approx\; 1 + 39.76\%$$
$$(1 + r_{0,t})^{(1/t)} \;=\; \sqrt[t]{1 + r_{0,t}} \;=\; (1 + r_{\overline{t}})$$

Confirm with your calculator that $r_{\overline{3}} \approx 39.76\%$,

$$1.3976 \;\cdot\; 1.3976 \;\cdot\; 1.3976 \;\approx\; (1 + 173\%)$$
$$(1 + r_{\overline{3}}) \cdot (1 + r_{\overline{3}}) \cdot (1 + r_{\overline{3}}) \;=\; (1 + r_{0,3})$$

In sum, if you invested money at a rate of 39.76% per annum for 3 years, you would end up with a total three-year holding rate of return of 173%. As is the case here, for very long periods, the order of magnitude of the annualized rate will often be so different from the holding rate that you will intuitively immediately register whether the quantity $r_{0,3}$ or $r_{\overline{3}}$ is meant. In the real world, very few rates of return, especially over long horizons, are quoted as holding rates of return. Almost all rates are quoted in annualized terms instead.

IMPORTANT

> The total holding rate of return over t years, called $r_{0,t}$, is translated into an annualized rate of return, called $r_{\overline{t}}$, by taking the t-th root:
>
> $$(1 + r_{\overline{t}}) \;=\; \sqrt[t]{1 + r_{0,t}} \;=\; (1 + r_{0,t})^{1/t}$$
>
> Compounding the annualized rate of return over t years yields the total holding rate of return.

Translating long-term dollar returns into annualized rates of return.

You also will often need to compute annualized rates of return from payoffs yourself. For example, what annualized rate of return would you expect from a \$100 investment today that promises a return of \$240 in 30 years? The first step is computing the total holding rate of return. Take the ending value (\$240) minus your beginning value (\$100), and divide by the beginning value. Thus, the total 30-year holding rate of return is

$$r_{0,30} \;=\; \frac{\$240 - \$100}{\$100} \;=\; 140\%$$
$$r_{0,30} \;=\; \frac{C_{30} - C_0}{C_0}$$

The annualized rate of return is the rate $r_{\overline{30}}$, which, if compounded for 30 years, offers a 140% rate of return,

$$(1 + r_{\overline{30}})^{30} \;=\; (1 + 140\%)$$
$$(1 + r_{\overline{t}})^t \;=\; (1 + r_{0,t})$$

Solve this by taking the 30th root,

$$(1+r_{\overline{30}}) \quad = (1+140\%)^{1/30} \quad = \sqrt[30]{1+140\%} \quad \approx 1+2.96\%$$
$$(1+r_{\overline{t}}) \quad = \quad (1+r_{0,t})^{1/t} \quad = \quad \sqrt[t]{1+r_{0,t}}$$

Subtracting 1, you see that a return of $240 in 30 years for an initial $100 investment is equivalent to a 2.96% annualized rate of return.

Compounding ≈ adding. Annualizing ≈ averaging.

In the context of rates of return, compounding is similar to adding, while annualizing is similar to averaging. If you earn 1% twice, your compounded rate is 2.01%, similar to the rates themselves added (2%). Your annualized rate of return is 1%, similar to the average rate of return of 2.01%/2 = 1.005%. The difference is the interest on the interest.

Averaging can lead to surprising results—returns that are much higher than what you earned per year.

Now assume that you have an investment that doubles in value in the first year and then falls back to its original value. What would its average rate of return be? Doubling from, say, $100 to $200 is a rate of return of +100%. Falling back to $100 is a rate of return of ($100–$200)/$200 = –50%. Therefore, the average rate of return would be [+100% + (–50%)]/2 = +25%.

But you have not made any money! You started with $100 and ended up with $100. If you compound the returns, you get the answer of 0% that you were intuitively expecting:

Look how deceptive!

$$(1+100\%) \cdot (1-50\%) \quad = \quad 1+0\% \quad \Rightarrow r_{0,2} \quad = 0\%$$
$$(1+r_{0,1}) \quad \cdot \ (1+r_{1,2}) \quad = \quad (1+r_{0,2})$$

It follows that the annualized rate of return $r_{\overline{2}}$ is also 0%. Conversely, an investment that produces +20% followed by –20% has an average rate of return of 0% but leaves you with a loss:

$$(1+20\%) \cdot (1-20\%) \quad = \quad (1-4\%) \quad \Rightarrow r_{0,2} \quad = -4\%$$
$$(1+r_{0,1}) \quad \cdot \ (1+r_{1,2}) \quad = \quad (1+r_{0,2})$$

For every $100 of your original investment, you now have only $96. The average rate of return of 0% does not reflect this loss. Both the compounded and therefore the annualized rates of return do tell you that you had a loss:

$$1+r_{\overline{2}} = \sqrt{(1+r_{0,2})} = \sqrt{1-4\%} = 1-2.02\% \Rightarrow r_{\overline{2}} \approx -2.02\%$$

➤ More about how arithmetic returns are “too high” in normally-distributed stock returns, Pg.143.

If you were an investment advisor and quoting your historical performance, would you rather quote your average historical rate of return or your annualized rate of return? (Hint: The industry standard is to quote the average rate of return, not the annualized rate of return!)

Do it!

Make sure to solve the following questions to gain more experience with compounding and annualizing over different time horizons.

Q 5.5. If you earn a rate of return of 5% over 4 months, what is the annualized rate of return?

Q 5.6. Assume that the two-year holding rate of return is 40%. The average (arithmetic) rate of return is therefore 20% per year. What is the annualized (geometric) rate of return? Is the annualized rate the same as the average rate?

Q 5.7. Is the compounded rate of return higher or lower than the sum of the individual rates of return? Is the annualized rate of return higher or lower than the average of the individual rates of return? Why?

Q 5.8. Return to Question 5.3. What was the annualized rate of return on the S&P 500 over the six years in the table?

Q 5.9. If the total holding interest rate is 50% for a five-year investment, what is the annualized rate of return?

Q 5.10. If the per-year interest rate is 10% for each of the next 5 years, what is the annualized five-year rate of return?

Duration and Maturity

We sometimes need summary statistics of how long a bond or a project will last. Maturity is the date of the very last payment of a bond. However, you would probably not consider a bond that pays \$1 in one year and \$1 in 30 years a 30-year bond—it's more like a 15-year bond. The **duration** can be calculated as

$$\text{Duration}(C_1 = \$1, C_{30} = \$1) = \frac{1 \times \$1 + 2 \times \$0 + \ldots + 29 \times \$0 + 30 \times \$1}{\$1 + \$0 + \ldots + \$1} = 15.5 \text{ years}$$

$$\text{Duration}(\{C_t\}) = \sum_t t/W \times C_t/$$

where W is the sum of all payments, $W = \sum_t C_t$. Intuitively, this bond has about a 15-year duration. Also intuitively, a bond that pays \$100 in one year and \$1 in 30 years has a shorter duration,

$$\text{Duration}(C_1 = \$100, C_{30} = \$1) = \frac{1 \times \$100 + 30 \times \$1}{\$100 + \$1} \approx 1.287$$

A zero bond has a duration equal to its maturity. One important variation is the **Macauley Duration**, which essentially replaces the raw cash flow with its present value. The intent is to discount far-away payouts more, which thus further shortens duration. For example, using our December 2015 yield curve,

$$\text{Duration}(C_1 = \$1, C_{100} = \$1) = \frac{1 \times \$100/1.0065 + 99 \times \$1/1.0301^{30}}{\$100/1.0065 + \$1/1.0301^{30}} \approx 1.004$$

There are also other measures of duration, but they are all basically summary statistics of when your "average" cash flow is going to occur.

Present Values with Time-Varying Interest Rates

The PV formula still looks very similar.

Let's proceed now to net present value with time-varying interest rates. What do you need to learn about the role of time-varying interest rates when computing NPV? The answer is essentially nothing new. You already know everything you need to know here. The net present value formula is still

$$\begin{aligned}
\text{NPV} &= \text{PV}(C_0) + \text{PV}(C_1) + \text{PV}(C_2) + \text{PV}(C_3) + \cdots \\
&= C_0 + \frac{C_1}{1 + r_{0,1}} + \frac{C_2}{1 + r_{0,2}} + \frac{C_3}{1 + r_{0,3}} + \cdots \\
&= C_0 + \frac{C_1}{1 + r_{\bar{1}}} + \frac{C_2}{(1 + r_{\bar{2}})^2} + \frac{C_3}{(1 + r_{\bar{3}})^3} + \cdots \\
&= C_0 + \frac{C_1}{1 + r_{0,1}} + \frac{C_2}{(1 + r_{0,1}) \cdot (1 + r_{1,2})} + \frac{C_3}{(1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3})} + \cdots
\end{aligned}$$

The only novelty is that you need to be more careful with your subscripts. You cannot simply assume that the multiyear holding returns (e.g., $1 + r_{0,2}$) are the squared 1-year rates of return ($(1 + r_{0,1})^2$). Instead, you must work with time-dependent costs of capital (interest rates). That's it.

Present values are still alike and thus can be added, subtracted, compared, and so on.

For example, say you have a project with an initial investment of \$12 that pays \$10 in one year and \$8 in five years. Assume that the 1-year interest rate is 5% and the five-year annualized interest rate is 6% per annum. In this case,

$$\text{PV}(\$10 \text{ in 1 year}) = \frac{\$10}{1.05} \approx \$9.52$$

$$\text{PV}(\$8 \text{ in 5 years}) = \frac{\$8}{1.06^5} \approx \$5.98$$

It follows that the project's total value *today* (time 0) is \$15.50. If the project costs \$12, its net present value is

$$\text{NPV} = -\$12 + \frac{\$10}{1.05} + \frac{\$8}{1.06^5} \approx \$3.50$$

$$\text{NPV} = C_0 + \frac{C_1}{1 + r_{0,1}} + \frac{C_5}{1 + r_{0,5}} = \text{NPV}$$

You can also rework a more involved project, similar to that in Exhibit 2.3,. But to make it more interesting, let's now use a hypothetical current term structure of interest rates that is upward-sloping. Assume this project requires an appropriate annualized discount rate of 5% over one year, and 0.5% more for every subsequent year, so that the cost of capital reaches 7% annualized in the fifth year. The valuation method works the same way as it did in Exhibit 2.3— you only have to be a little more careful with the interest rate subscripts. The project's value is thus

Here is a typical NPV example.

➤ Hypothetical Project Cash Flows, Exhibit 2.3, Pg.28.

Time	Project Cash Flow	Annualized Rate	Com-pounded	Discount Factor	Value
t	C_t	$r_{\bar{t}}$	$r_{0,t}$	$\frac{1}{1+r_{0,t}}$	$\text{PV}(C_t)$
Today	–\$900	any		1.0000	–\$900.00
Y1	+\$200	5.0%	5.0%	0.9524	\$190.48
Y2	+\$200	5.5%	11.3%	0.8985	\$179.69
Y3	+\$400	6.0%	19.1%	0.8396	\$335.85
Y4	+\$400	6.5%	28.6%	0.7773	\$310.93
Y5	–\$100	7.0%	40.3%	0.7130	–\$71.30
			Net Present Value (Sum):		\$45.65

Q 5.11. A project costs \$200 and will provide cash flows of +\$100, +\$300, and +\$500 in consecutive years. The annualized interest rate is 3% per annum over one year, 4% per annum over two years, and 4.5% per annum over three years. What is this project's NPV?

5.2 Inflation

Inflation is the increase in the price of the same good.

Let's make our world even more realistic—and complex—by working out the effects of inflation. **Inflation** is the process by which the same good costs more in the future than it does today. With inflation, the price level is rising and thus money is losing its value. For example, if inflation is 100%, an apple that costs $0.50 today will cost $1 next year, a banana that costs $2 today will cost $4, and corporate finance textbooks that cost $200 today will cost $400.

Inflation matters when contracts are not written to adjust for it.

Inflation may or may not matter in a corporate context, depending on how contracts are written. If you ignore inflation and write a contract that promises to deliver bread for the price of $1 next year, it is said to be in **nominal terms**—and you may have made a big mistake. The money you will be paid will be worth only half as much. You will only be able to buy one apple for each loaf of bread that you had agreed to sell for $1, not the two apples that anyone else will enjoy. On the other hand, you could write your contract in **real terms** (or **inflation-indexed terms**) today, in which case the inflationary price change would not matter. That is, you could build into your promised banana delivery price the inflation rate from today to next year. An example would be a contract that promises to deliver bananas at the rate of four apples per banana. If a contract is indexed to inflation, then inflation does not matter. However, in the United States inflation often does matter, because most contracts are in nominal terms and not inflation-indexed. Therefore, you have to learn how to work with inflation. What effect, then, does inflation have on returns? On (net) present values? This is our next subject.

Measuring the Inflation Rate

The CPI is the most common inflation measure.

The first important question is how you should define the inflation rate. Is the rate of change of the price of apples the best measure of inflation? What if apples (the fruit) become more expensive, but Apples (the computers) become less expensive? Defining inflation is actually rather tricky. To solve this problem, economists have invented *baskets* or *bundles* of goods and services that are deemed to be representative. Economists then measure an average price change for these items. The official source of most inflation measures is the **Bureau of Labor Statistics** (**BLS**), which determines the compositions of a number of common bundles (indexes) and publishes the average total price of these bundles on a monthly basis. The most prominent such inflation measure is a hypothetical bundle of average household consumption, called the **Consumer Price Index** (**CPI**). (The CPI components are roughly: housing 40%, food 20%, transportation 15%, medical care 10%, clothing 5%, entertainment 5%, others 5%.) The BLS offers inflation data at http://www.bls.gov/cpi/, and the *Wall Street Journal* prints the percent change in the CPI at the end of its regular column "Money Rates."

From Dec 2014 to Dec 2015, the inflation rate was a remarkably low 0.7% per annum. (And there were some months with negative inflation rates, too—called **deflation**!)

Year	2010	2011	2012	2013	2014	2015
CPI	1.5	3.0	1.7	1.5	0.8	0.7

A number of other indexes are also commonly used as inflation measures, such as the **Producer Price Index** (**PPI**) or the broader **GDP Deflator**. They typically move very similarly to the CPI. Over short periods, one expects these rates to move fairly close together (on average, not every month); but over longer periods, they can diverge. There are also more specialized bundles, such as computer or flash memory chip inflation indexes (their prices usually decline), or price indexes for particular regions.

The CPI matters—even if it is calculated incorrectly.

The official inflation rate is not just a number that mirrors reality—it is important in itself, because many contracts are specifically indexed to a particular inflation definition. For example, even if actual true inflation is zero, if the officially reported CPI rate is positive, the government

The German Hyperinflation of 1922

Many economists now believe that a modest inflation rate between 1% and 3% per year is a healthy number. It's not so easy to maintain.

The Roman Emperor Gallienus is still infamous two-thousand years later for having changed the value of the Roman Denarius by reducing its silver content by a factor of 100.

The most famous episode of extreme inflation (**hyperinflation**) occurred in Germany from August 1922 to November 1923. Prices more than quadrupled every month. The price for goods was higher in the evening than in the morning! Stamps had to be overprinted by the day, and shoppers went out in the morning with bags of money that were worthless by the end of the day. By the time Germany printed 1,000 billion Mark Bank Notes, no one trusted Marks anymore. This hyperinflation was stopped only by a drastic currency and financial system reform. But high inflation is not just a historic artifact. In 2015, Venezuela had an inflation rate of 180%. Don't trust Venezuelan Bolivars!

Yet recent experience has humbled us economists (further) by proving that it can also be difficult to push *up* inflation. In the **Great Recession** (the financial crisis of 2008-11), the Fed tried to fuel inflation by pushing money into the hands of consumers. The idea was to get them to lose just a little trust in the currency and spend it, so as to raise the value of much underwater real estate. But consumers turned around and deposited the money back into their banks, which in turn redeposited the money with—you guessed it—the Fed.

must pay out more to Social Security recipients. The lower the official inflation rate, the less the government has to pay. You would therefore think that the government has the incentive to understate inflation. But, strangely, this has not been the case. On the contrary, there are strong political interest groups that hinder the BLS from even fixing mistakes that everyone knows overstate the CPI—that is, corrections that would result in *lower* official inflation numbers. In 1996, the Boskin Commission, consisting of a number of eminent economists, found that the CPI overstates inflation by about 74 basis points per annum—a huge difference. The main reasons were and continue to be that the BLS has been tardy in recognizing the growing importance of such factors as effective price declines in the computer and telecommunications industries and the role of superstores such as Wal-Mart and Target.

Before we get moving, a final warning:

IMPORTANT

The common statement "in today's dollars" can be ambiguous. Most people mean "inflation-adjusted." Some people mean present values (i.e., "compared to an investment in risk-free bonds"). When in doubt, ask!

Q 5.12. Read the Bureau of Labor Statistics' website descriptions of the CPI and the PPI. How does the CPI differ conceptually from the PPI? Are the two official rates different right now?

Real and Nominal Interest Rates

Nominal returns are what is normally quoted. Real returns are adjusted for inflation. They are what you want to know if you want to consume.

To work with inflation and to learn how you would properly index a contract for inflation, you first need to learn the difference between a **nominal return** and a **real return**. The nominal rate is what is usually quoted—a return that has not been adjusted for inflation. In contrast, the real rate of return "somehow takes out" inflation from the nominal rate in order to calculate a return "as if" there had been no price inflation to begin with. The real return reflects the fact that, in the presence of inflation, dollars in the future will have less purchasing power than dollars today.

It measures your trade-off between present and future consumption, taking into account the change in prices.

An extreme 100% inflation rate example: Prices double every year.

Start with a simple exaggerated scenario: Assume that the inflation rate is 100% per year and you can buy a bond that promises a *nominal* interest rate of 700%. What is your *real* rate of return? To find out, assume that \$1 buys one apple today. With an inflation rate of 100%, you need \$2 next year to buy the same apple. Your gross return will be $\$1 \cdot (1 + 700\%) = \$8$ for today's \$1 of investment. But this \$8 will apply to apples costing \$2 each. Your \$8 will buy 4 apples, not 8 apples. Your real rate of return (1 apple yields 4 apples) is therefore

$$r_{real} = \frac{(4 \text{ Apples for } \$8) - (1 \text{ Apple for } \$2)}{(1 \text{ Apple for } \$2)} = 300\%$$

For each dollar invested today, you will be able to buy only 300% more apples next year (not 700% more) than you could buy today. This is because inflation will reduce the purchasing power of your dollar by more than one half.

Here is the correct conversion formula from nominal to real rates.

The correct formula to adjust for the inflation rate (π) is again a "one-plus" type formula. In our example, it is

$$(1 + 700\%) = (1 + 300\%) \cdot (1 + 100\%)$$
$$(1 + r_{nominal}) = (1 + r_{real}) \cdot (1 + \pi)$$

Turning this formula around gives you the real rate of return,

$$(1 + r_{real}) = \frac{1 + 700\%}{1 + 100\%} = 1 + 300\%$$
$$(1 + r_{real}) = \frac{(1 + r_{nominal})}{(1 + \pi)}$$

In plain English, a nominal interest rate of 700% is the same as a real interest rate of 300%, given an inflation rate of 100%.

IMPORTANT

The relation between the nominal rate of return ($r_{nominal}$), the real rate of return (r_{real}), and the inflation rate (π) is

$$(1 + r_{nominal}) = (1 + r_{real}) \cdot (1 + \pi) \qquad (5.2)$$

For small rates, adding/subtracting is an okay approximation.

As with compounding, if the rates are small, the mistake of just subtracting the inflation rate from the nominal interest rate to obtain the real interest rate is not too grave. For example, with our (30-year) 3% Treasury, if inflation were to remain 0.7%, the correct real interest rate would be 2.28% and not 2.30%:

➤ Bills, notes, and bonds, Sect. 5.3, Pg.86.

$$(1 + 3\%) \approx (1 + 2.28...\%) \cdot (1 + 0.7\%) \approx 1 + 2.28...\% + 0.7\% + 0.0002.....$$
$$(1 + r_{nominal}) = (1 + r_{real}) \cdot (1 + \pi) = 1 + r_{real} + \pi + \underbrace{r_{real} \cdot \pi}_{\text{cross-term}}$$

The difference between the correct and the approximation, i.e., the cross-term, is trivial, and easily swamped by measurement noise in the current inflation rate and uncertainty about future inflation rates. However, when inflation and interest rates are high—as they were, for example, in the United States in the late 1970s—then the cross-term can be important.

➤ Adding or compounding interest rates, and the cross-term, Pg.20.

A positive time value of money—that the same amount of money today is worth more than tomorrow—is only true for nominal quantities, not for real quantities. Only nominal interest rates are (usually) not negative. In the presence of inflation, real interest rates not only *can* be negative, but often *are* negative. For example, in December 2015, the one-month Treasury paid 0.14% while the inflation rate was 0.7%, implying a real interest rate of about –0.56%. Every dollar you invested in such U.S. Treasuries would be worth *less* in real purchasing power one month later. You would have ended up with more cash—but also with *less* purchasing power.

Real interest rates can be negative.

Gold

Sometimes, the price of gold has been used as a measure of inflation. Although gold is not a great measure of purchasing power in general, it does make it easy to conduct long-run comparisons. A Roman legionary was paid the equivalent of 2.31 oz of gold a year. A U.S. Army private nowadays is paid the equivalent of 11.01 oz—about five times as much. A Roman centurion was paid 38.58 oz. A U.S. army captain is paid 27.8 oz, about a quarter less. Thus, as an asset class, an investment in gold would have earned a rate of return roughly in line with the income growth over two millenia. *Erb and Harvey (2013)*

Q 5.13. From memory, write down the relationship between nominal rates of return (r_{nominal}), real rates of return (r_{real}), and the inflation rate (π).

Q 5.14. The nominal interest rate is 20%. Inflation is 5%. What is the real interest rate?

Inflation in Net Present Values

When it comes to inflation and net present value, there is a simple rule: Never mix apples and oranges. The beauty of NPV is that every project's cash flows are translated into the same units: today's dollars. Keep everything in the same units in the presence of inflation, so that this NPV advantage is not lost. When you use the NPV formula, always discount nominal cash flows with nominal discount rates, and real (inflation-adjusted) cash flows with real (inflation-adjusted) discount rates.

The most fundamental rule: never mix apples and oranges. Nominal cash flows must be discounted with nominal interest rates.

Let's return to our "apple" example. With 700% nominal interest rates and 100% inflation, the real interest rate is $(1 + 700\%)/(1 + 100\%) - 1 = 300\%$. What is the value of a project that gives 12 apples next year, given that apples cost \$1 each today and \$2 each next year?

Our example discounted both in real and nominal terms.

There are two methods you can use:

1. Discount the nominal cash flow of 12 apples next year ($\$2 \cdot 12 = \$24$) with the nominal interest rate. Thus, the 12 future apples are worth

$$\frac{\text{Nominal Cash Flow}}{1 + \text{Nominal Rate}} = \frac{\$24}{1 + 700\%} = \$3$$

Discount nominal cash flows with nominal rates. Discount real cash flows with real rates.

2. Discount the real cash flows of 12 apples next year with the real interest rate. Thus, the 12 future apples are worth

$$\frac{\text{Real Cash Flow}}{1 + \text{Real Rate}} = \frac{12 \text{ Apples}}{1 + 300\%} = 3 \text{ Apples}$$

in today's apples. Because an apple costs $1 today, the 12 apples next year are worth $3 today.

Both the real and the nominal methods arrive at the same NPV result. The opportunity cost of capital is that if you invest one apple today, you can quadruple your apple holdings by next year. Thus, a 12-apple harvest next year is worth 3 apples to you today. The higher nominal interest rates already reflect the fact that nominal cash flows next year are worth less than they are this year. As simple as this may sound, I have seen corporations first work out the real value of their goods in the future, and then discount this with standard nominal interest rates. Just don't!

IMPORTANT

- Discount nominal cash flows with nominal interest rates.
- Discount real cash flows with real interest rates.

Either works. Never discount nominal cash flows with real interest rates, or vice-versa.

Usually, it is best to work only with nominal quantities.

If you want to see this in algebra, the reason that the two methods come to the same result is that the inflation rate cancels out,

$$\text{PV} = \frac{\$24}{1 + 700\%} = \frac{12\text{A}}{1 + 300\%} = \frac{12\text{A} \cdot (1 + 100\%)}{(1 + 300\%) \cdot (1 + 100\%)}$$
$$= \frac{\text{N}}{1 + r_{\text{nominal}}} = \frac{\text{R}}{1 + r_{\text{real}}} = \frac{\text{R} \cdot (1 + \pi)}{(1 + r_{\text{real}}) \cdot (1 + \pi)}$$

where N is the nominal cash flow, r is the real cash flow, and π is the inflation rate. Most of the time, it is easier to work in nominal quantities. Nominal interest rates are far more common than real interest rates, and you can simply use published inflation rates to adjust the future price of goods to obtain future expected nominal cash flows.

Q 5.15. If the real interest is 3% per annum and the inflation rate is 8% per annum, then what is the present value of a $500,000 nominal payment next year?

5.3 U.S. Treasuries and the Yield Curve

The simplest and most important benchmark bonds nowadays are Treasuries. They have known and certain payouts.

It is now time to talk in more detail about the most important financial market in the world today: the market for bonds issued by the U.S. government. These bonds are called Treasuries and are perhaps the simplest projects around. This is because, in theory, Treasuries cannot fail to pay. They promise to pay U.S. dollars, and the U.S.-controlled Federal Reserve has the right to print more U.S. dollars if it were ever to run out. Thus, there is no uncertainty about repayment for Treasuries. (In contrast, some European countries or U.S. states that borrow in currencies that they cannot create may well not have the money to pay and therefore default.)

U.S. Treasury bills, notes, and bonds have different maturities.

The shorthand "Treasury" comes from the fact that the debt itself is issued by the U.S. Treasury Department. There are three main types:

1. **Treasury bills** (often abbreviated as **T-bills**) have maturities of up to one year.
2. **Treasury notes** have maturities between one and ten years.
3. **Treasury bonds** have maturities greater than ten years.

The 30-year bond is often called the **long bond**. Together, the three are usually just called **Treasuries**. Conceptually, there is really no difference among them. All are really just obligations issued by the U.S. Treasury. Indeed, there can be Treasury notes today that are due in 3 months—such as a 9-year Treasury note that was issued 8 years and 9 months ago. This is really the same obligation as a 3-month Treasury bill that was just issued. Thus, we shall be (very) casual with name distinctions.

Magnitudes

In late 2015, the United States Federal Government owed over $18 trillion in Treasury obligations (on a GDP of about $17.5 trillion). With a population of 322 million, this debt translated to over $55,000 per person. With 125 million households and 121 million full-time workers, it represented about $150,000 per household or worker. (A worse problem, however, is that the United States has already promised benefits to future retirees that far exceed this number.) But the United States also has assets. It owns more than $100 trillion in land, infrastructure, and mineral rights under the land.

The Treasuries market is one of the most important financial markets in the world.

After Treasuries are sold by the government, they are then actively traded in what is one of the most important financial markets in the world today. It would not be uncommon for a dedicated bond trader to buy $100 million of a Treasury note originally issued 10 years ago that has 5 years remaining, and 10 seconds later sell $120 million of a three-year Treasury note issued 6 years ago. Large buyers and sellers of Treasuries are easily found, and transaction costs are very low. Trading volume is huge: Around 2015, it was about $500 billion per trading day. Therefore, the annual trading volume in U.S. Treasuries—about 252 · $500 billion ≈ $130 trillion, where 252 is the approximate number of trading days per year—was about an order of magnitude larger than the annual U.S. gross domestic product (GDP).

Who owns them?

Who owns them? About $6 trillion is owed to foreigners, with the Chinese (our largest creditor) holding about $1.2 trillion of our bonds. about $12 trillion is owed to ourselves. The U.S. Federal Reserve estimated that domestically about 20% were held by individuals, 25% by banks and mutual funds, 15% by public and private pension funds, 15% by state and local governments, and 25% by other investors.

The yield curve shows the annualized interest rate as a function of bond maturity.

Interest rates on Treasuries change every moment, depending on their maturity terms. Fortunately, you already know how to handle time-varying rates of return, so we can now put your knowledge to the test. The principal tool for working with Treasury bonds is the **yield curve** (or **term structure of interest rates** or just the **term structure**). It is a graphical representation, where the time to maturity is on the x-axis and the annualized interest rates are on the y-axis. There are also yield curves on non-Treasury bonds, but the Treasury yield curve is so prominent that unless clarified further, the yield curve should be assumed to mean investments in U.S. Treasuries. (A more precise name would be the "U.S. Treasuries yield curve.") This yield curve is so important that most other debts in the financial markets, like mortgage rates or bank lending rates, are "benchmarked" relative to the Treasury yield curve. For example, if your firm wants to issue a five-year bond, your creditors will want to compare your interest rate to that offered by equivalent Treasuries, and often will even describe your bond as offering "x basis points above the equivalent Treasury."

Q 5.16. What are the three types of Treasuries? How do they differ?

Yield Curve Shapes

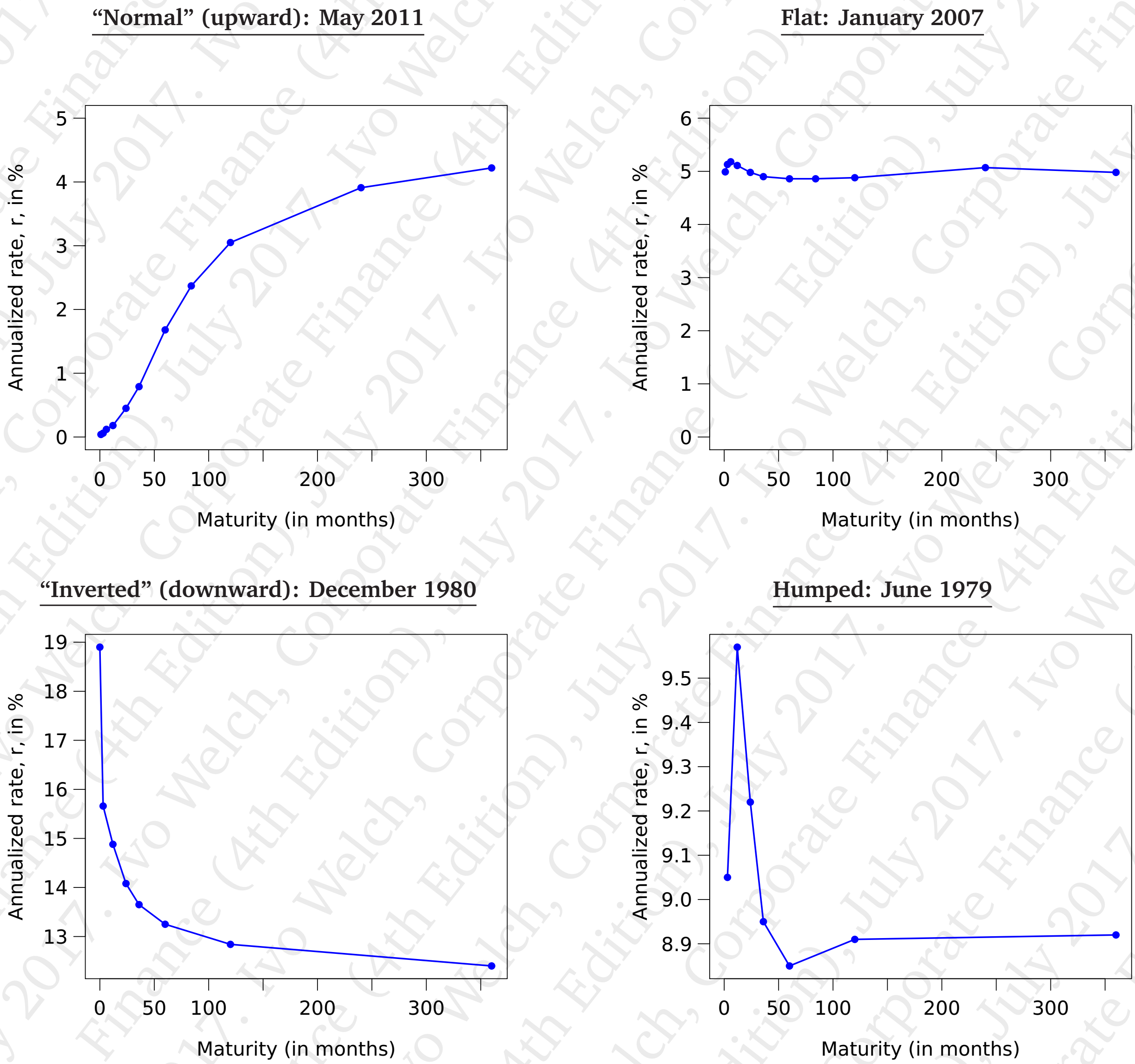


Exhibit 5.1: *Various Historical Yield-Curve Shapes.* The upward slope is so common that it is considered the "normal" shape. In 2016, the yield curve was normal, but not as steep as that in May 2011. A downward slope is sometimes called "inverted."

Exhibit 5.1 shows some historical yield curves. They are commonly classified into four basic shapes:

Yield curves are often, but not always, upward-sloping.

1. **Flat:** There is little or no difference between annualized short-term and long-term rates. A flat yield curve is basically the scenario that was the subject of the previous chapter. It means you can simplify $(1+r_{0,t}) \approx (1+r)^t$.
2. **Upward-sloping ("normal"):** Short-term rates are lower than long-term rates. This is the most common shape. It means that longer-term interest rates are higher than shorter-term interest rates. Since 1934, the steepest yield curve (the biggest difference between the long-term and the short-term Treasury rates) occurred in October 1992, when the long-term interest rate was 7.3% and the short-term interest rate was 2.9%—just as the economy pulled out of the recession of 1991. As of 2016, the yield curve has been upward-sloping since the Great Recession.
3. **Downward-sloping ("inverted"):** Short-term rates are higher than long-term rates.
4. **Humped:** Short-term rates and long-term rates are lower than medium-term rates.

Inverted and humped yield curves are relatively rare.

Macroeconomic Implications of Different Yield Curve Shapes

Economists and pundits have long wondered what they can learn from the shape of the yield curve about the future of the economy. It appears that the yield curve shape is a useful—though unreliable and noisy—signal of where the economy is heading. Steep yield curves often signal emergence from a recession, as you will see in Exhibit 5.4 on Page 96. Inverted yield curves often signal an impending recession. But can't the Federal Reserve Bank control the yield curve and thereby control the economy? It is true that the Fed can influence the yield curve—and since 2008, it has worked on influencing it like never before. But ultimately the Fed does not control it—instead, it is the broader demand and supply for savings and credit in the economy that determines it. Economic research has shown that the Fed typically has a good deal of influence on the short end of the Treasury curve—by expanding and contracting the supply of money and short-term loans in the economy—but not much influence on the long end of the Treasury curve, especially in the long-run. And even in the financial crisis of 2008, the Fed's influence on the short end was ultimately limited, too—the nominal rate already stood at 0% and there was little the Fed could do to drop it further. (Large negative nominal rates are not possible.) In fact, by flooding the economy with cheap money, the Fed was trying to push banks to lend and people to spend money—but people instead just deposited the cash right back into the banks!

If you want to undertake your own research, you can find historical interest rates at the St. Louis Federal Reserve Bank at http://research.stlouisfed.org/fred. There are also the Treasury Management Pages at http://www.tmpages.com/. Or you can look at SmartMoney.com for historical yield curves. PiperJaffray.com has the current yield curve—as do many other financial sites and newspapers. Finance.yahoo.com/bonds provides not only the Treasury yield curve but also yield curves for many other types of bonds.

Common data sources for interest rates.

An Example: The Yield Curve on December 31, 2015

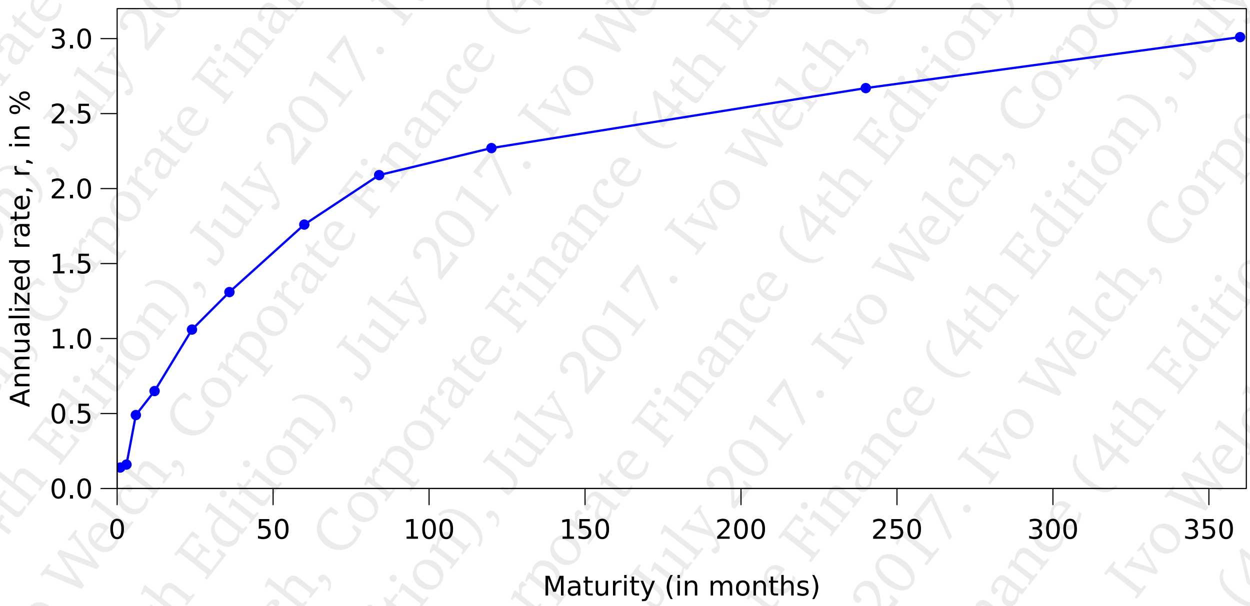
Let's focus on working with one particular yield curve. Exhibit 5.2 shows the Treasury yields on December 31, 2015. This yield curve had the most common upward slope. The curve tells you that if you had bought a 3-month Treasury at the end of the day on December 31, 2015, your annualized interest rate would have been 0.16% per annum. (A \$100 investment would turn into $\$100 \cdot (1+0.16\%)^{1/4} \approx \$100 \cdot 1.0003998 \approx \$100.04$ on March 31, 2016.) If you had

We will analyze the Treasury yield curve at the end of December 2015.

bought a thiry-year bond, your annualized interest rate would have been 3.01% per annum, or \$243.43 on December 2045 for a \$100 investment.

You can interpolate annualized interest rates on the yield curve.

Sometimes it is necessary to determine an interest rate for a bond that is not listed. This is usually done by interpolation. For example, if you had wanted to find the yield for a 4-year bond, a reasonable guess would have been an interest rate halfway between the 3-year bond and the 5-year bond. In December 2015, this would have been an annualized yield of $(1.31\% + 1.76\%)/2 \approx 1.54\%$. (This is *not* exact, as you can guess by noticing that the yield curve looks more concave.)



	⟵ Months			Years ⟶							
	1	3	6	1	2	3	5	7	10	20	30
12/31/2015	0.14	0.16	0.49	0.65	1.06	1.31	1.76	2.09	2.27	2.67	3.01

Exhibit 5.2: *The Treasury Yield Curve on December 31, 2015.* These rates are annualized yields to maturity (internal rates of return) calculated from Treasury prices. If they were truly Treasury zero-bonds, they would just be the standard discount rates computed from the final payment and today's price, but we ignore the details here. Such yield curves can be found on many websites. The yield curve changes every day—although day-to-day changes are usually small. Our example works primarily with this particular yield curve. Source: Federal Reserve, www.federalreserve.gov/releases/h15/data.htm.

The December 2015 yield curve was upward-sloping: Annualized interest rates were higher for longer maturities.

As notation for the annualized horizon-dependent interest rates, we continue using our earlier method. We call the two-year annualized interest rate $r_{\overline{2}}$ (here, 1.06%), the three-year

Deeper: There are some small inaccuracies in my description of yield curve computations. My main simplification is that U.S. yield curves are based on semi-annually-compounded coupon bonds in real life, whereas our textbook pretends that the yield is quoted on a zero bond. In corporate finance, the yield difference between annual compounding and semi-annual compounding is almost always inconsequential. However, if you want to become a fixed-income trader, you cannot take this approximation literally. Consult a dedicated fixed-income text instead.

annualized interest rate $r_{\overline{3}}$ (here, 1.31%), and so on. It is always these overlined-subscript yields that are graphed in yield curves. Let's work with this particular yield curve, assuming it is based exclusively on zero-bonds, so you don't have to worry about interim payments.

Holding rates of return First, let's figure out how much money you will have at maturity. That is, how much does an investment of \$500,000 in U.S. two-year notes (i.e., a loan to the U.S. government of \$500,000) on December 31, 2015, return on December 31, 2017? Use the data in Exhibit 5.2. Because the yield curve prints annualized rates of return, the total two-year holding rate of return (as in Formula 5.1) is the twice compounded annualized rate of return,

Computing the holding rate of return for 2-year and 30-year Treasuries.

➤ Annualizing, Formula 5.1, Pg.78.

$$r_{0,2} = 1.0106 \cdot 1.0106 - 1 \approx 2.13\%$$
$$r_{0,2} = (1 + r_{\overline{2}}) \cdot (1 + r_{\overline{2}}) - 1$$

so your \$500,000 will turn into

$$C_2 \approx (1 + 2.13\%) \cdot \$500{,}000 \approx \$510{,}656$$
$$C_2 = (1 + r_{0,2}) \cdot C_0$$

on December 31, 2017. (In the real world, you might have to pay a commission to arrange this transaction, so you would end up with a little less.)

What if you invest \$500,000 into 30-year Treasuries? Your 30-year holding rate of return is

$$r_{0,30} = 1.0301^{30} - 1 \approx 2.434 - 1 \approx 143.4\%$$
$$r_{0,30} = (1 + r_{\overline{30}})^{30} - 1$$

Thus, an investment of $C_0 = \$500{,}000$ in December 2015 turns into a return of $C_{30} \approx \$1.2$ million by December 2045.

Forward rates of return Second, let's figure out what the yield curve in December 2015 implied about the 1-year interest rate from December *2016* to December 2017. This would be best named $r_{1,2}$. It is an interest rate that begins in one year and ends in two years. As already mentioned, this is called the *forward rate*.

Let's work out one forward rate implied by the December 2015 yield curve.

The 1-year annualized interest rate is $r_{\overline{1}} = 0.65\%$. The two-year annualized rate of return is $r_{\overline{2}} = 1.06\%$. You already know that you can work out the two holding rates of return, $r_{0,1} = 0.65\%$ and $r_{0,2} = (1 + r_{\overline{2}})^2 - 1 \approx 1.0213\%$. You only need to use the compounding formula to determine $r_{1,2}$:

$$(1 + 2.13\%) = (1 + 0.65\%) \cdot (1 + r_{1,2}) \Rightarrow r_{1,2} \approx 1.47\%$$
$$(1 + r_{0,2}) = (1 + r_{0,1}) \cdot (1 + r_{1,2})$$

Note that this forward rate $r_{1,2}$ is higher than both $r_{\overline{1}}$ and $r_{\overline{2}}$ from which you computed it.

Exhibit 5.3 summarizes our two-year calculations, and extends them by another year. (This helps you to check your results in an exercise below.) One question you should ask yourself is whether I use so many subscripts in the notation just because I enjoy torturing you. The answer is an emphatic no: The subscripts are there for good reason. When you look at Exhibit 5.3, for example, you have to distinguish between the following:

Is the proliferation of subscripts torture or necessity?

- the three holding rates of return, $r_{0,t}$ (0.65%, 2.13%, and 3.98%)
- the three annualized rates of return, $r_{\overline{t}}$ (0.65%, 1.06%, and 1.31%)

Maturity	Total Holding	Rates of Return Annualized	Compounded Rates
1 Year	$(1+0.65\%)$	$=(1+0.65\%)^1$	$=(1+0.65\%)$
	$(1+r_{0,1})$	$=(1+\bar{r}_1)^1$	$=(1+r_{0,1})$
2 Years	$(1+2.13\%)$	$\approx(1+1.06\%)^2$	$\approx(1+0.65\%)\cdot(1+1.47\%)$
	$(1+r_{0,2})$	$=(1+\bar{r}_2)^2$	$=(1+r_{0,1})\cdot(1+r_{1,2})$
3 Years	$(1+3.98\%)$	$\approx(1+1.31\%)^3$	$\approx(1+0.65\%)\cdot(1+1.47\%)\cdot(1+1.81\%)$
	$(1+r_{0,3})$	$=(1+\bar{r}_3)^3$	$=(1+r_{0,1})\cdot(1+r_{1,2})\cdot(1+r_{2,3})$

Exhibit 5.3: *Relation between Holding Returns, Annualized Returns, and Year-by-Year Returns on December 31, 2015, by Formula.* The individually compounded rates are the future interest rates. They are implied by the annualized rates quoted in the middle column. The text worked out the two-year case. You will work out the three-year case in Question 5.17.

- the three individual annual rates of return $r_{t-1,t}$ (0.65%, 1.47%, and 1.81%), where the second and third forward rates begin at different moments in the future.

In real life, you have not just three yearly Treasuries, but many Treasuries between 1 day and 30 years. Anyone dealing with Treasuries (or CDs or any other fixed-income investment) that can have different maturities or start in the future must be prepared to suffer double subscripts.

Yes, corporate projects have double subscripts, too!

If Treasuries offer different annualized rates of return over different horizons, do corporate projects have to do so, too? Almost surely yes. If nothing else, they compete with Treasury bonds for investors' money. And just like Treasury bonds, many corporate projects do not begin immediately, but may take a year or more to prepare. Such project rates of return are essentially forward rates of return. Double subscripts—yikes, but sometimes there is no way out of painful notation in the real world!

Q 5.17. Compute the three-year holding rate of return on December 31, 2015. Then, using the two-year holding rate of return on December 31, 2015, and your calculated three-year holding rate of return, compute the forward interest rate for a 1-year investment beginning on December 31, *2017*, and ending on December 31, 2018. Are these the numbers in Exhibit 5.3?

Q 5.18. Repeat the calculation with the five-year annualized rate of return of 1.76%. That is, what is the five-year holding rate of return, and how can you compute the *annualized* forward interest rate for a *two*-year investment beginning on December 31, *2018*, and ending on December 31, 2020?

Bond Payoffs and Your Investment Horizon

Your investment horizon has no link to the time patterns of bond payoffs you invest in. You can always sell long-term bonds to get money quickly, if need be.

Should there be a link between your personal investment horizon and the kinds of bonds you may be holding? Let's say that you want to buy a three-year zero-coupon bond because it offers 1.31%, which is more than the 0.65% that a 1-year zero-coupon bond offers—but you also want to consume in one year. Can you still buy the longer-term bond? There is good news and bad news. The good news is that the answer is yes: There is no link whatsoever between your desire to get your money back and consume, and the point in time when your three-year bond pays off. You can always buy a three-year bond today, and sell it before maturity, such as next year when it will have become a two-year bond. The bad news is that in our perfect and certain market, this investment strategy will still only get you the 0.65% that the 1-year bond offers. If you buy \$100 of the three-year bond for $P = \$100/1.0131^3 \approx \$96.17$ today, next year it will be a two-year bond with an interest rate of 1.47% in the first year and 1.81% in the second year (both worked out in Exhibit 5.3). You can sell this bond next year for a price of

$$\frac{\$100}{1+r_{1,3}} = \frac{\$100}{(1+r_{1,2})\cdot(1+r_{2,3})} = \frac{\$100}{1.0147\cdot 1.0181} \approx \$96.80$$

Your 1-year holding rate of return would therefore be only (\$96.80 – \$96.17)/ \$96.17 ≈ 0.65%—the same rate of return you would have received if you had bought a 1-year bond to begin with. There is no free lunch here.

The Effect of Interest Rate Changes on Short-Term and Long-Term Bonds

Treasuries pay what they promise. They have no default risk. They do have the risk of interim interest rate changes.

Are long-term bonds riskier than short-term bonds? Of course, recall that repayment is no less certain with long-term Treasury bonds than short-term Treasury notes. (This would be an issue of concern if you were to evaluate corporate projects that can go bankrupt. Long-term corporate bonds are often riskier than short-term corporate bonds—most firms are unlikely to go bankrupt this week, but more likely to go bankrupt over a multidecade time horizon.) So, for Treasury bonds, as long as Congress does not go crazy, there should be no uncertainty as far as payment uncertainty is concerned. But there may still be some interim risk of a different kind; and even though we have not yet fully covered it, you can still intuitively figure out why this is so. Ask yourself how economy-wide bond prices (interest rates) can change in the interim (before maturity). What are the effects of sudden interest rate changes before maturity on bond values? It turns out that an equal-sized interest rate movement can be much more dramatic for long-term bonds than for short-term bonds. Let me try to illustrate why.

First, the effect of a 10 bp yield change on the price of a 30-year bond.

The 30-year bond: Work out the value of a \$1,000 30-year zero-bond at the 3.01% interest rate prevailing. It costs $\$1{,}000/1.0301^{30} \approx \$410.79$. You already know that when prevailing interest rates go up, the prices of outstanding bonds drop and you will lose money. For example, if interest rates increase by 10 basis points to 3.11%, the bond value decreases to $\$1{,}000/1.0311^{30} \approx \$399.00$. If interest rates decrease by 10 basis points to 2.91%, the bond value increases to $\$1{,}000/1.0291^{30} \approx \$422.93$. Thus, the effect of a 10-basis-point change in the prevailing 30-year yield induces an immediate percent change (an instant rate of return) in the value V of your bond of

$$\text{Up 10 bp:} \quad r = \frac{V(r_{\overline{30}} = 3.11\%) - V(r_{\overline{30}} = 3.01\%)}{V(r_{\overline{30}} = 3.01\%)} \approx \frac{\$399.00 - \$410.79}{\$410.79} \approx -2.87\%$$

$$\text{Down 10 bp:} \quad r = \frac{V(r_{\overline{30}} = 2.91\%) - V(r_{\overline{30}} = 3.01\%)}{V(r_{\overline{30}} = 3.01\%)} \approx \frac{\$422.93 - \$410.79}{\$410.79} \approx +2.96\%$$

So for every \$1 million you invest in 30-year bonds, you expose yourself to about \$30,000 in instant risk for every 10-basis-point yield change in the economy.

Second, the effect of a 10 bp point change on the price of a 1-year note.

The 1-year note: To keep the example identical, let's now assume that the 1-year note also has an interest rate of 3.01% and consider the same 10-basis-point change in the prevailing interest rate. In this case, the equivalent computations for the value of a 1-year note are \$970.78 at 3.01%, \$971.72 at 2.91%, and \$969.84 at 3.11%. Therefore, the equivalent instant rates of return are

$$\text{Up 10 bp:} \quad r = \frac{V(r_{\bar{1}} = 3.11\%) - V(r_{\bar{1}} = 3.01\%)}{V(r_{\bar{1}} = 3.01\%)} \approx \frac{\$971.72 - \$970.78}{\$970.78} \approx -0.097\%$$

$$\text{Down 10 bp:} \quad r = \frac{V(r_{\bar{1}} = 3.11\%) - V(r_{\bar{1}} = 3.01\%)}{V(r_{\bar{1}} = 3.01\%)} \approx \frac{\$969.84 - \$970.78}{\$970.78} \approx +0.097\%$$

For every \$1 million you invest in 1-year notes, you expose yourself to about \$1,000 risk in instant risk for every 10-basis-point yield change in the economy.

An equal interest rate move affects longer-term bonds more strongly.

It follows that the value effect of an *equal-sized* change in prevailing interest rates is more severe for longer-term bonds. In turn, it follows that if the bond is due tomorrow, interest rate changes can usually wreak very little havoc. You will be able to reinvest tomorrow at whatever the new rate will be. A long-term bond, on the other hand, may lose (or gain) a lot of value.

Again, in the interim, T-bonds are not risk-free!

In sum, you should always remember that Treasury bonds are risk-free in the sense that they cannot default (fail to return the promised payments), but they are risky in the sense that interim interest changes can alter their values. Only the most short-term Treasury bills (say, due overnight) can truly be considered risk-free—virtually everything else suffers interest-rate change risk.

IMPORTANT

> Though "fixed income," even Treasuries do not guarantee a "fixed rate of return" over horizons shorter than their maturities. Day to day, long-term Treasury bonds are generally riskier investments than short-term Treasury bills, because interest-rate changes have more impact on them.

For illustration, I have ignored volatility of changes and earned interest.

Confession time: I have pulled two cheap tricks on you. First, in the real world, what if short-term, economy-wide interest rates typically experienced yield shifts of plus or minus 100 basis points, while long-term, economy-wide interest rates never budged? If this were true, long-term bonds could even be safer than short-term bonds. However, the empirical evidence from 1990 to 2016 suggests that day-to-day changes of both were of similar magnitude—about plus or minus 5 basis points a day. (One-month yields changed by about 7 basis points a day.) Second, I ignored that between today and tomorrow, you would also earn 1 day of interest. On a \$1,000,000 investment in 1-years, this would be about \$25; in 30-years, about \$120. Thus about \$100 should be added to the long-term bond investment strategy—but \$100 on a \$30,000 risk exposure was small enough to keep ignorance bliss.

Q 5.19. A ten-year and a 1-year zero-bond both offer an interest rate of 8% per annum.

1. How does an increase of 1 basis point in the prevailing interest rate change the value of the 1-year bond? (Use 5 decimals in your calculation.)
2. How does an increase of 1 basis point in the prevailing interest rate change the value of the ten-year bond?
3. What is the ratio of the value change over the interest change? (In calculus, this would be called the derivative of the value with respect to interest rate changes.) Which derivative is larger?

A Tragic Error: "Paper Losses"?!

If you really need cash from a bond investment in 20 years, doesn't a prevailing interest rate increase cause only an interim **paper loss**? This is a capital logical error many investors commit. Say that a 10-basis-point increase happened overnight, and you had invested $1 million yesterday. You would have lost $30,000 of your net worth in 1 day! Put differently, waiting 1 day would have saved you $30,000 or allowed you to buy the same item for $30,000 less. Paper money is actual wealth. Thinking paper losses are any different from actual losses is a common but capital error. (The only exception to this rule is that realized gains and losses have different tax implications than unrealized gains and losses.) Avoiding this conceptual mistake is more important than learning any formulas in this book.

"Only" a paper loss: A cardinal error!

➤ Tax treatment of realized and unrealized capital gains, Sect. 11.4, Pg.263.

"Paper losses" are no less real than realized losses.

IMPORTANT

5.4 Why Does the Yield Curve Usually Slope Up?

Aren't you already wondering *why* the yield curve is not usually flat? Take our example yield curve from December 2015. Why did the 30-year Treasury bonds in December 2015 pay 3.01% per year, while the one-month Treasury bills paid only 0.14% per year? And why is the upward slope the most common shape?

But why? Why? Why?

First, let's look at the historical data. We cannot easily visualize the entire historical yield curve in a two-dimensional graph, but we can plot the historical yields on the short-term 3-month Treasury and, say, the 20-year Treasury. Exhibit 5.4 shows that the spread between them ranged from negative (though usually only briefly) to about 300 bp per year. (Humped shapes are rare, so if the long-term rate is below the short-term rate, the yield curve is most likely inverted.) The plot shows that inverted shapes occurred often just before a recession. Since the Great Recession of 2008, the Fed has kept short-term rates close to zero, which has resulted in unusually large spreads to the long-term Treasury. Long-term rates have been coming down.

Historical Yield Curves

To understand these dynamics better, let's work with a simpler two-year example. Let's say that the yield curve tells you that the 1-year rate is $r_{\bar{1}} = 5\%$ and the two-year rate is $r_{\bar{2}} = 10\%$. You can work out that the 1-year forward rate is then $r_{1,2} \approx 15.24\%$. There are really only two possible explanations:

The two possible explanations are (1) higher future interest rates and/or (2) compensation for risk.

1. The 1-year interest rate next year will be higher than the 5% that it is today. Indeed, maybe next year's 1-year interest rate will be the 15.24% that it would be in a perfect world with perfect certainty. In equation form, is it the case that $(1+r_{1,2})\cdot(1+E(r_{2,3})) = (1+r_{1,3})$, so $E(r_{2,3}) = (1+r_{1,3})/(1+r_{1,2})-1$? (I am using "E" as an abbreviation for "expected"—which you will learn about in the next chapter.)
2. Investors tend to earn higher rates of return holding long-term bonds than they do holding short-term bonds. For example, if the yield curve were to remain at exactly the same shape next year, then a $100 investment in consecutive 1-year bonds would give you interest of only about $10.25, while the same investment in two-year bonds would give you (on average) $21. Is $E(r_{2,3}) < (1+r_{1,3})/(1+r_{1,2})-1$?

In other words, the question is whether higher long-term interest rates today predict higher interest rates in the future, or whether they offer extra compensation for investors willing to hold longer-term bonds. Let's consider two possible variants of each of these two possibilities.

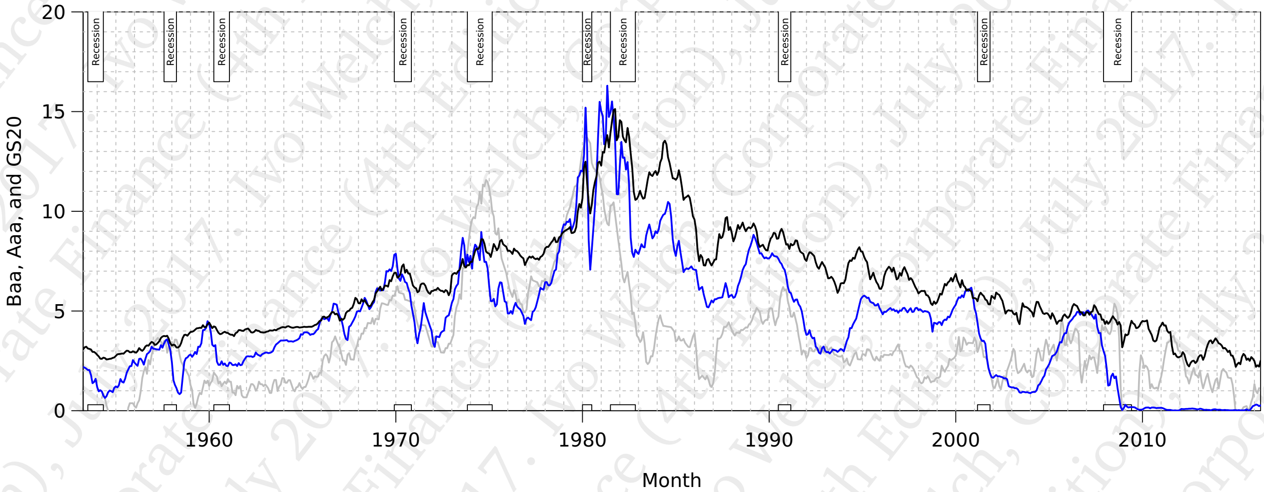


Exhibit 5.4: *3-Month and 20-year Treasury Yields.* The blue line is the yield on 3-month Treasury bills; the black line is the yield on 20-year Treasury bonds. When the blue line was above the black line, the yield curve was inverted. The gray line is the 12-month moving average inflation rate. Recessions are marked at the top (and compressed at the bottom). The data comes from the Federal Reserve FRED database.

Does the Yield Curve Predict Higher Future Inflation?

If inflation is high, investors (typically) demand higher interest rates.

In general, when inflation is higher, you would expect investors to demand higher nominal interest rates. Consequently, you would expect nominal rates to go up when inflation rate expectations are going up. Similarly, you would expect nominal rates to go down when inflation rate expectations are going down. Of course, demand and supply do not mean that real rates of return need to be positive—indeed, the real rate of return is often negative, but the alternative of storing money under the mattress is even worse.

Are higher future inflation rates the cause of higher future interest rates?

Therefore, our first potential explanation for an upward-sloping yield curve is that investors believe that cash will be worth progressively less in the more distant future. That is, even though you will be able to earn higher interest rates over the long run, you may also believe that the inflation rate will increase from today's rate. Because inflation erodes the value of higher interest rates, interest rates should then be higher in the future just to compensate you for the lesser value of money in the future. Of course, this argument would apply only to a yield curve computed from Treasury debt that pays off in nominal terms. It should not apply to any bond payoffs that are inflation-indexed.

TIPS are inflation-indexed Treasury bonds. They are not affected by inflation.

Fortunately, since 1997 the Treasury has been selling bonds that are inflation-indexed. These bond contracts are written so that they pay out the promised interest rate plus the CPI inflation rate. They are called Treasury Inflation Protected Securities (**TIPS**), or sometimes just **CPI bonds**. By definition, they should not be affected by inflation in a perfect market. If the nominal yield curve is upward-sloping because of higher future inflation rates, then a TIPS-based real yield curve should not be upward-sloping.

Inflation-Neutral Bonds

As it turns out, inflation-adjusted bonds had already been invented once before! The world's first known inflation-indexed bonds were issued by the Commonwealth of Massachusetts in 1780 during the Revolutionary War. These bonds were invented to deal with severe wartime inflation and discontent among soldiers in the U.S. Army with the decline in the purchasing power of their pay. Although the bonds were successful, the concept of indexed bonds was abandoned after the immediate extreme inflationary environment passed, and largely forgotten. In 1780, the bonds were viewed as an irregular expedient, because there was no formulated economic theory to justify indexation.

Robert Shiller, "The Invention of Inflation-Indexed Bonds in Early America," October 2003

Conveniently, the Treasury website also shows a TIPS-based yield curve. They were

Inflation-adjusted bond prices suggest expectations of inflation were not the main driver of the upward-sloping yield curve.

➤ Inflation Adjusting, Formula 5.2, Pg.84.

Maturity	5-year	7-year	10-year	20-year	30-year
TIPS Interest Rate	0.45%	0.59%	0.73%	1.07%	1.28%

Recall that for small figures, the difference between the nominal and the real rate is about the inflation rate.

Maturity	5-year	7-year	10-year	20-year	30-year
Ordinary Treasury Bonds	1.76%	2.09%	2.27%	2.67%	3.01%
TIPS Interest Rate	0.45%	0.59%	0.73%	1.07%	1.28%
Implied Inflation	1.31%	1.50%	1.54%	1.60%	1.73%

Without going into more details, the implied inflation rate contains a little bit of risk compensation, too, and more for longer-term projects. Thus, differences in inflation expectations can explain *at most* an 0.4% difference between 5-year and 30-year nominal interest rates, and likely quite a bit less. Say 0.25% is a good guestimate. This 0.25% is only about one-fifth part of the 1.25% spread in nominal interest rates. Trust me that these numbers have also been reasonably representative for the last few decades of U.S. history. We can conclude that, even if increasing inflation expectations can play a minority role, they were not the main reason for the common upward nominal yield curve slope.

Q 5.20. In June, 2016, an inflation-adjusted 30-year Treasury bond offered a real yield of about 0.7% per year. The equivalent non-inflation-adjusted bond offered 2.25% per year. In what inflation scenario would you be better off buying one or the other? (The most recent historical inflation rate was 1% per year.)

Does The Yield Curve Predict Higher Future Interest Rates?

Does a high forward interest rate predict a high future interest rate?

A closely related possibility is that the yield curve is typically upward-sloping because short-term interest rates will be higher in the future. This is more generic than the previous explanation—higher future interest rates need not be caused by higher future inflation expectations. Maybe the 30-year yield of 3.01% was much higher than the 1-year yield of 0.65% because investors expected the 1-year interest rate in 2044 to be much much higher than 3% (the forward rate, $r_{29,30}$). This does not tell you *why* investors would expect interest rates to be so much higher in 2044 than in 2015—maybe they expect that capital will be more scarce then and investment opportunities will be better—but we can speculate about this even if we do not know the precise reason.

Alas, the historical data tells us "probably not much."

Unfortunately, we do not have a direct estimate of future interest rates the way we had a direct estimate of future inflation rates (from TIPS). Therefore, investigating this hypothesis requires looking at many years of evidence to learn whether future interest rates were well predicted by prevailing forward rates. The details are beyond our scope. However, I can tell you the punchline: Expectations of higher future rates of return are not the reason why the yield curve is typically upward-sloping (except maybe at the very short end of the yield curve, say, for interest rates that are for cash investments for less than 1 month).

Does The Yield Curve Identify Bargains?

It must be either higher future interest rates or higher compensation for long-term bond investors.

If it is not the case that future interest rates are higher when forward rates are higher, it means that we are dealing with the second possible reason: On average, investors must have earned more in long-term bonds than in rolled-over short-term bonds. The empirical data confirms that you would have ended up historically with more money if you had bought 30-year bonds than if you had bought one-month bonds every month for 30 years.

Different Preferences?

One reason why this may have been the case is the habitat theory: Different investors may only like different types of bonds, to the point that different-maturity bond markets are segmented. The fact that there are habitats may well be true, but it is not so clear why long-term bond investors would then be so scarce that they require more compensation, while short-term bond investors would be so abundant that bills can be sold for higher prices. And if this were the only reason, then why would borrowers not always prefer to sell short-term bonds instead? And why would other investors not try to step in and get rich (by "arbitraging" the difference)?

Free money? Not in a perfect market.

So why were long-term bonds better investments than short-term bonds? Maybe the yield curve was upward-sloping because investors were stupid. In this case, you might conclude that the 30-year bond offering 3.01% was a much better deal than the 1-year bond offering 0.65%. Alas, investor stupidity seems highly unlikely as a good explanation. The market for Treasury bond investments is close to perfect in the sense that we have used the definition. It is very competitive. If there was a great deal to be had, thousands of traders would have immediately jumped on it. More likely, the interest rate differential does not overthrow the old tried-and-true axiom, *You get what you pay for*. It is just a fact of life that investments for which the payments are tied down to occur in 30 years must offer higher interest rates now in order to entice investors—for some good reason yet to be identified. Again, it is important that you keep in mind that your cash and consumption are *not* tied down if you invest in a 30-year bond, because you can, of course, sell your 30-year bond tomorrow to another investor if you so desire.

Does It Compensate Investors For Risk?

The answer is probably compensation for risk.

➤ Bond Risk, Sect. 5.3, Pg.93.

If it isn't market stupidity that allows you to earn more money in long-term bonds than in rolled-over short-term bonds, then what else could it be? The empirical evidence suggests that it is most likely the phenomenon explained in Section 5.3: Interim changes in prevailing interest rates have much more impact on long-term bonds than they have on short-term bonds. Recall that rolling over short-term bonds insulates you from the risk that interest rates will change in the future. If you hold a one-day bond and interest rates double by tomorrow, you can just purchase more bonds tomorrow that will offer you twice the interest rate. In contrast, if you hold a long-term bond, you could lose your shirt if interest rates go up in the future. With long-term bonds being riskier than short-term bonds, investors only seem to want to buy them if they get some extra rate of return. Otherwise, they prefer rolling short bonds. Thus, long-term bonds need to offer investors more return on average than short-term bonds.

5.5 Corporate Time-Varying Costs of Capital

Extend this insight to corporations: Longer-term projects, even if they are not more likely to default, often face a higher cost of capital, and therefore should have to deliver higher returns.

Now that you understand that the yield curve is usually upward-sloping for a good reason, you should recognize the family resemblance: Corporate projects are offering cash flows, just like Treasury bonds. Thus, it should not surprise you that longer-term projects usually have to offer higher rates of return than shorter-term projects. And just because a longer-term project offers a higher expected rate of return does not necessarily mean that it has a higher NPV. Conversely, just because shorter-term borrowing allows firms to pay a lower expected rate of return does not necessarily mean that this creates value. (Neither firms nor the U.S. Treasury rely exclusively on short-term borrowing.) Paying a higher expected rate of return for longer-term obligations is (usually) a fact of life.

IMPORTANT

Even in a perfect market without uncertainty:

- The appropriate cost of capital (rate of return) should usually depend on how long-term the project is.
- The term structure is usually upward-sloping. Short-term corporate projects usually have lower costs of capital than long-term projects.
- Conversely, corporations usually face lower costs of capital (expected rates of return offered to creditors) if they borrow short term rather than long term.

The difference between long-term and short-term rates is called the **Term Premium**.

Time-Varying Expected Rates of Return vs. Time- and Risk-Varying Expected Rates of Return.

Let me give you a short preview now. In Chapter 10, you will learn about the CAPM. The CAPM is the most common model used to discount future cash flows in NPV applications. It is a model that relates your project's required expected rate of return to its risk. In practice, the CAPM allows you to use higher (risk-free) rates of return for cash flows farther in the future. Thus, the CAPM has one term that is (more or less reflective of) the term-premium and one term for the risk-premium. In this sense, it can be viewed as a generalization of the point of this chapter that longer-term projects usually require higher (opportunity) costs of capital. If the second term is zero, then all you have left is the term premium.

Basic Usage and Reasonable guesstimate.

It turns out that the first term (the term premium) has worked much better than the second (the risk premium). Thus, in real life, it is often more important for you to understand that you usually need to increase your cost-of-capital estimate for longer-term cash flows, than it is for you to understand the much more complex second CAPM term.

Alas, there is one second-order complication: the term-premium for corporate cash flows can be different from the term-premium for Treasuries. Although this is true, the first-order approximation is usually that the two are similar. That is, if a 20-year Treasury offers an expected rate of return that is 2% higher than that of a 1-year Treasury, you should probably guess that a corporate cash flow in 20 years should offer an expected rate of return that is also about 2% higher than that of its 1-year counterpart. (The second-order corrections depend on much deeper, more difficult, and harder-to-prove reasoning and will most likely gross up or shade down the Treasury spread only by a little bit.)

Summary

This chapter covered the following major points:

- Different horizon investments can offer different rates of return. This phenomenon is often called time-varying rates of return.
- The general formula for compounding works just as well for time-varying rates of return as it does for time-constant rates of return. You only lose the ability to exponentiate (one plus the 1-year rate of return) when you want to compute multiyear rates of return.
- A holding rate of return can be annualized for easier interpretation.
- The graph of annualized interest rates as a function of maturity is called the "term structure of interest rates" or the "yield curve."
- The yield curve is usually upward-sloping. However, no law of finance is violated if it is downward-sloping (inverted), humped, or flat.
- Net present value also works just as well for time-varying interest rates. You merely need to use the appropriate rate of return as the opportunity cost of capital in the denominator.
- An important side observation: "Paper losses" are no different from real losses.
- Inflation is the process by which money will buy fewer goods in the future than it does today. If contracts are inflation-indexed in a perfect market, inflation is irrelevant. This is rarely the case.
- The relationship between nominal interest rates, real interest rates, and inflation rates is

$$(1 + r_{\text{nominal}}) = (1 + r_{\text{real}}) \cdot (1 + \text{Inflation Rate})$$

- Unlike nominal interest rates, real interest rates can—and often have been—negative.
- In NPV, you can either discount real cash flows with real interest rates, or discount nominal cash flows with nominal interest rates. The latter is usually more convenient.
- TIPS are Treasury bonds that protect against future inflation. Short-term bond buyers are also less exposed to inflation rate changes than long-term bond buyers.
- Higher long-term interest rates could be due either to expectations of higher future interest rates or to extra required compensation for investors willing to hold longer-term bonds. The empirical evidence suggests that historically the latter has been the more important factor.
- Corporations should realize that corporate project cash flows need to be discounted with specific costs of capital that may depend on when the cash flows will come due. It is not unusual that cash flows in the more distant future require higher discount rates.

Preview of the Chapter Appendix in the Companion

The appendix to this chapter explains

- how you can translate the yield curve (in Exhibit 5.3) step by step into forward and total holding rates of return.
- how shorting works in the real world, and how you can lock in a future interest today with clever bond transactions today.
- (again) how the "duration" of bonds helps you measure when you receive your cash flows on average.
- continuous compounding, which is a different way of quoting interest rates.
- that Treasury notes and bonds are not really zero bonds (as we pretended in this chapter) but coupon bonds, and why this rarely matters in a corporate context. True Treasury zero bonds are called STRIPS.

Keywords

Answers

Q 5.1 $r_{0,2} = (1 + r_{0,1}) \cdot (1 + r_{1,2}) - 1 = 1.02 \cdot 1.03 - 1 = 5.06\%$

Q 5.2 Solve $(1 + x) \cdot (1 + 22\%) = (1 - 50\%)$, so the project had a rate of return of –59.00%.

Q 5.3 The first three-year compounded rate of return was $r_{2010,2012} \approx (1 + 0.150) \cdot (1 + 0.021) \cdot (1 + 0.16) - 1 \approx +36.2\%$. (The notation is a bit ambiguous when month and day are omitted, because the first return is from the end of 2009 to the end of 2010.) The second three-year rate was $r_{2013,2015} \approx +52.42\%$. The full six-year compounded rate of return was thus $r_{2010,2015} \approx (1{+}.362\%) \cdot (1{+}.524\%) - 1 \approx +107.6\%$. Although these were very fat years for stock investors. the realized rate was indeed time-varying.

Q 5.4 The returns were (–33%, +50%, –67%, +100%). Thus the average rate was 12.5% and the overall rate of return was –33.33%. It is always true that the compound rate of return is always less than the average rate of return. The example shows that the two can differ in sign.

Q 5.5 $1.05^{12/4} \approx 15.76\%$

Q 5.6 The annualized rate of return is $\sqrt{1.4} - 1 \approx 18.32\%$. It is therefore lower than the 20% average rate of return.

Q 5.7 The compounded rate of return is always higher than the sum, because you earn interest on interest. The annualized rate of return is lower than the average rate of return, again because you earn interest on the interest. For example, an investment of \$100 that turns into an investment of \$200 in two years has a total holding rate of return of 100%—which is an average rate of return of $100\%/2 = 50\%$ and an annualized rate of return of $\sqrt{(1 + 100\%)} - 1 \approx 41.42\%$. Investing \$100 at 41% per annum would yield \$200, which is lower than 50% per annum.

Q 5.8 The six-year holding rate of return was 107.6%. Thus, the annualized rate of return was $r_{\overline{6}} = \sqrt[6]{1 + 107.6\%} - 1 \approx 12.9\%$.

Q 5.9 $r_{0,5} = 50\% \quad (1 + r_{\overline{5}})^5 = 1.50 \implies r_{\overline{5}} = 1.50^{1/5} - 1 \approx 8.45\%$.

Q 5.10 The annualized five-year rate of return is the same 10%.

Q 5.11 This project is worth

$$-\$200 + \frac{\$100}{1.03} + \frac{\$300}{1.04^2} + \frac{\$500}{1.045^3} \approx \$612.60$$

Q 5.12 The CPI is the average price change to the consumer for a specific basket of goods. The PPI measures the price that producers are paying. Taxes, distribution costs, government subsidies, and basket composition drive a wedge between these two inflation measures.

Q 5.13 $(1 + r_{\text{nominal}}) = (1 + r_{\text{real}}) \cdot (1 + \pi)$

Q 5.14 $1.20/1.05 \approx 1.1429$. The real interest rate is 14.29%.

Q 5.15 The nominal interest rate is $1.03 \cdot 1.08 - 1 = 11.24\%$. Therefore, the cash flow is worth about $\$500{,}000/1.1124 \approx \$449{,}479$.

Q 5.16 Bills, notes, and bonds. T-bills have maturities of less than 1 year. T-notes have maturities from 1 to 10 years. T-bonds have maturities greater than 10 years.

Q 5.17 Yes. The answers are right in the table. The three-year rate of return is $1.0131^3 - 1 \approx 3.98\%$. The forward rate is $1.0398/(1.0065 \cdot 1.0147) - 1 \approx 1.81\%$.

Q 5.18 $r_{0,5} = 1.0176^5 - 1 \approx 9.12\%$. Therefore, $1 + r_{3,5} = 1.0176^5/1.0131^3 - 1 \approx 4.94\%$. This is a two-year forward holding rate of return. Thus, it is $1.0494^{(1/2)} - 1 \approx 2.44\%$ in annualized terms.

Q 5.19 (1) For the 1-year bond, the value of a \$100 bond changes from $\$100/1.0800 \approx \$92.59259$ to $\$100/1.0801 \approx \$92.58402$. This is about a –0.009% change. (2) For the ten-year bond, the value of a \$100 bond changes from $\$100/1.08^{10} \approx \$46.31935$ to $\$100/1.0801^{10} \approx \$46.27648$. This is a –0.09% change—ten times that of the 1-year bond. (3) The derivative of the 1-year bond is $-0.009/0.01 = -0.9 \approx -1$. The derivative of the ten-year bond is $-0.09/0.01 \approx -9$. The derivative of the ten-year bond is about nine times more negative.

Q 5.20 If the inflation rate will increase to more than $1.0225/1.007 - 1 \approx 1.5\%$ per year, the inflation-adjusted bond will be better. Otherwise, the non-inflation adjusted bond will be better.

End of Chapter Problems

Q 5.21. Are you better off if a project first returns –10% followed by +30%, or if it first returns +30% followed by –10%?

Q 5.22. Compare two stocks. Both have earned 8% per year on average. However, stock A has oscillated between 6% and 10%. Stock B has oscillated between 3% and 13%. (For simplicity, say that they alternated.) If you had bought $500 in each stock, how much would you have had 10 years later?

Q 5.23. (Strange) Stock A always alternated between +20% and –10% in the past. Stock B earned 4.5% per annum.

1. What was the average rate of return for stock A?
2. What was the average rate of return for stock B?
3. On a 1-year basis, would a risk-neutral investor prefer +20% or –10% with equal probability, or 4.5% for sure?
4. How much would each dollar invested 10 years ago in stock A have earned?
5. How much would each dollar invested 10 years ago in stock B have earned?
6. What is going on here?

Q 5.24. Return to Question 5.3. What was the annualized geometric rate of return, and what was the average rate of return on the S&P 500? Would stock brokers prefer to tell their clients the former or the latter?

Q 5.25. On June 23, 2016, the Brits voted to exit the EU. The following were the daily values of an investment (in a fund called SPY):

June	27	28
Dollars	199.60	203.20

If returns were to accumulate at the same rate over an entire year (252 trading days), what would a $100 investment turn into?

Q 5.26. If the annualized five-year rate of return is 10%, what is the total five-year holding rate of return?

Q 5.27. If the annualized five-year rate of return is 10%, and if the first year's rate of return is 15%, and if the returns in all other years are equal, what are they?

Q 5.28. The annual interest rate from year t to year $t+1$ is $r_{t,t+1} = 5\% + 0.3\% \cdot t$ (e.g., the rate of return from year 5 to year 6 is $5\% + 0.3\% \cdot 5 = 6.5\%$).

1. What is the holding rate of return of a ten-year investment today?
2. What is the annualized interest rate of this investment?

Q 5.29. A project has cash flows of +$100 (now at time 0), and –$100, +$100, and –$100 at the end of consecutive years. The interest rate is 6% per annum.

1. What is the project's NPV?
2. How does the value change if all cash flows will occur one year later?
3. Repeat these two questions, but assume that the 1-year (annualized) interest rate is 5%, the two-year is 6%, the three-year is 7%, the four-year is 8%, and so on.

Q 5.30. What is the current inflation rate?

Q 5.31. What is the annualized current nominal interest rate on 30-day U.S. Treasury bills?

Q 5.32. Using the information from Questions 5.30 and 5.31, compute the annualized current real interest rate on 30-day Treasuries.

Q 5.33. If the nominal interest rate is 7% per year and the inflation rate is 2% per year, what is the exact real rate of return?

Q 5.34. The inflation rate is 1.5% per year. The real rate of return is 2.0% per year. A perpetuity project that paid $100 this year will provide income that grows by the inflation rate. Show what this project is truly worth. Do this in both nominal and real terms. (Be clear on what *never* to do.)

Q 5.35. If the annualized rate of return on insured tax-exempt municipal bonds will be 3% per annum and the inflation rate remains at 2% per annum, then what will be their real rate of return over 30 years?

Q 5.36. If the real interest rate is –1% per annum and the inflation rate is 3% per annum, then what is the present value of a $1,000,000 nominal payment next year?

Q 5.37. Inflation is 2% per year; the interest rate is 8% per year. Your perpetuity project has cash flows that grow at 1% faster than inflation forever, starting with $20 next year.

1. What is the real interest rate, both accurate (the "1+" version) and approximate (the subtraction version)?
2. What is the correct project PV?
3. What would you get if you grew a perpetuity project of $20 by the real growth rate of 1%, and then discounted it at the nominal cost of capital?
4. What would you get if you grew a perpetuity project of $20 by the nominal growth rate of 3%, and then discounted it at the real cost of capital?

Performing either of the latter two calculations is not an uncommon mistake in practice.

Q 5.38. You must value a perpetual lease. It will cost $100,000 each year *in real terms*—that is, its proceeds will not grow in real terms, but just contractually keep pace with inflation. The prevailing interest rate is 8% per year, and the inflation rate is 2% per year forever. The first cash flow of your project *next year* is $100,000 *quoted in today's real dollars*. What is the PV of the project? (Warning: Watch the timing and amount of your first payment.)

Q 5.39. If the real rate of return has been about 1% per month for long-term bonds, what would be the value of an investment that costs $100 today and returned $200 in 10 years?

Q 5.40. At your own personal bank, what is the prevailing savings account interest rate?

Q 5.41. Look up today's yield curve on a financial website. What is the 1-year rate of return on a risk-free Treasury? What is the ten-year rate of return on a risk-free Treasury? What is the 30-year rate of return on a risk-free Treasury?

Q 5.42. The 1-year forward interest rates are

Y1	Y2	Y3	Y4	Y5	Y6
3%	4%	5%	6%	6%	6%
Y7	**Y8**	**Y9**	**Y10**	**Y11**	**Y12**
7%	7%	7%	6%	5%	4%

1. Compute the 12 n-year compounded holding rates of return from now to year n.
2. Compute the 12 annualized rates of return.
3. Draw the yield curve.
4. Is there anything wrong in this example?

Q 5.43. The *annualized* interest rates are

Y1	Y2	Y3	Y4	Y5	Y6
3%	4%	5%	6%	6%	6%
Y7	**Y8**	**Y9**	**Y10**	**Y11**	**Y12**
7%	7%	7%	6%	5%	4%

1. Draw the yield curve.
2. Compute the 12 n-year compounded holding rates of return from now to year n.
3. Compute the 12 1-year forward rates of return.
4. Is there anything wrong in this example?

Q 5.44. At today's prevailing Treasury rates, how much money would you receive from an investment of $100 in 1 year, 10 years, and 30 years? What are their annualized rates of return? What are their total holding rates of return?

Q 5.45. Do long-term bonds pay more than short-term bonds because you only get money after a long time—money that you could need earlier?

Q 5.46. A five-year, zero-coupon bond offers an interest rate of 8% per annum.

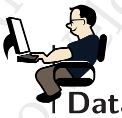
1. How does a 1-basis-point increase in the prevailing interest rate change the value of this bond in relative terms?
2. What is the ratio of the relative bond value change over the interest change? (This is the derivative of the value with respect to interest rate changes.)
3. How does the derivative of wealth with respect to the interest rate vary with the length of the bond?

Q 5.47. Look at this week's interest rate on ordinary T-bonds and on TIPS. (You should be able to find this information, e.g., in the *Wall Street Journal* or through a fund on the Vanguard website.) What is the implied inflation rate at various time horizons?

Q 5.48. The yield curve is usually upward-sloping. Assess whether this means that the following statements are true or false:

1. Investors earn a higher annualized rate of return from long-term T-bonds than short-term T-bills.
2. Long-term T-bonds are better investments than short-term T-bills.
3. Investors are expecting higher inflation in the future than they are today.
4. Investors who are willing to take the risk of investing in long-term bonds on average earn a higher rate of return because they are taking more risk (that in the interim bond prices fall / interest rates rise).

Q 5.49. Evaluate and Discuss: Does the evidence suggest that long-term bonds tend to earn higher average rates of return than short-term bonds? If yes, why is this the case? If no, why is this not possible?



Data and Programming for Masters Students

The St. Louis Federal Reserve Bank's "FRED" http://research.stlouisfed.org/fred offers the premier (and free!) data base of historical key interest rate (and more than 420,000 other economic time series), especially but not only for U.S. Treasury bonds. There are many other good and free similar bond related websites: the Treasury Management Pages at http://www.tmpages.com/, yahoo at https://finance.yahoo.com/bonds, Bloomberg at https://www.bloomberg.com/markets/rates-bonds/government-bonds/us, smartmoney at http://SmartMoney.com (nice for historical yield curves), etc. http://finance.yahoo.com/bonds provides not only the Treasury yield curve, but also yield curves for many other types of bonds. The WSJ at https://www.wsj.com and others offer Treasury strip (i.e., zero bond) data—just google around. (Note that it is often much easier to work with yield curves than with the individual trading bond prices, and easier to work with strip yield curves than with coupon yield curves.)

Task A: Graph (in 3-D) the duration of a bond as a function of its semi-annual coupon yield and its maturities. (No data is necessary.)

Task B: Assume that the Treasury zero-bond yield curve is $\log(Y+1)$. Thus, a 0-day has a yield of 0%, a 30-year has a yield of about 3.4%. Now price a set of Treasury coupon bonds that happens to have the same coupon as the zero-bond has yield. For example, the 10-year coupon bond would pay 2.3%/2 coupon semi-annually. What is the YTM for this coupon bond? Plot the yield curve based on zero bonds and, on the same graph, based on coupon bonds. Discuss how the two yield curves look like?

The remaining tasks require data from FRED. Download the historical yield curves at the end of each calendar year from Fred. That is, work only with annual frequency yield data in the following tasks:

Task C: Write a computer program that creates a 3-D plot for annual historical Treasury yields (z), with maturity on the x-axis and calendar year on the y-axis. Extra points if your program also automates the (FRED) data download, png creation, and posting on a web site.

Task D: Although not exact, it is possible to approximate bond rate of returns with yield data. Start with an example. Say you have a bond with 60 months left to maturity that quotes a YTM of 5.1%. Next year, this bond has 48 months left to maturity and, say, quotes a YTM of 4.0%. You can translate this into a percent price change. During this year, you would also receive the interest (reasonably assumable to be some number between 4.0% and 5.1%, though in real life this may be either accrued interest or coupon payments). Using the calculated price change and interest receipts, you can now approximate the bond's holding rate of return.

Now plot the bond rates of return on a 3-D plot, similar to the one in Task C, with maturity and calendar year on the x-axis and y-axis? Can you color losses red?

(Can you generalize your program to deal with various coupons, YTMs, and maturities?)

Task E: Using the yield on the 1-year bond ($r_{0,1}$) and the 2-year bond ($r_{0,2}$), calculate the forward rate ($f_{1,2}$). Now investigate how this forward rate relates to the past forward rate, the prevailing 1-year rate, and the future realized 1-year rate. For example, you could research whether the forward rate helps to predict the future 1-year rate (or whether is it primarily risk compensation) by running the regression $r_{1,2} - r_{0,1} = \alpha + \beta \cdot f_{1,2} - r_{0,1} + \epsilon$.

(Can you do this for other frequencies, too? Is the result always the same? Has it changed over the calendar years?)