Perpetuities
(Welch, Chapter 03-A)

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Maintained Assumptions

- We assume **perfect markets**:
  1. No differences in opinion.
  2. No taxes.
  3. No transaction costs.
  4. No big sellers/buyers—infinitely many clones that can buy or sell.
- We again assume **perfect certainty**, so we know what the RoR is on every project.
- We assume constant RoRs (per year).

General Questions

- Are there any shortcut NPV formulas for long-term projects—at least under certain common assumptions?
- Or, do we always have to compute long summations for projects with many, many periods?
- Why do some of the folks have the magic ability to quickly tell you estimates that would take you hours to figure out with the NPV formula?

Specific Sample Questions

- What is the value of a firm that generates \$1 million in earnings per year and grows by the inflation rate?
- If your firm earns \$5 million/year, and the interest rate is 5%, what is its approximate value?
- What is a Pro-Forma terminal market-value estimate?

Simple Perpetuities

A **Perpetuity** is a financial instrument that pays C dollars per period *forever*.

- If the interest rate is constant and the first payment from the perpetuity arrives in period 1,

$$PV(C, r) = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r} \ .$$

- This notation is *very common* in finance:
  - C and r are the two real input variables.
  - t is an ephemeral counter (not an input variable).

Perpetuity Footnotes

Make sure you know when the first cash flow begins: Tomorrow [t=1], not today [t=0]!

- I sometimes write C_1/r to remind myself of timing, even though cash flows are the same at time 1 as they are at time 25—I could have written C_{25} instead.

(NFL) Booth Review

Write out the formula $\sum_{t=1}^{\infty} \frac{C}{(1+r)^t}$:

Programming Language

$\sum_{t=1}^{T} f(t)$ is

```
function sum(integer T)
   sumup <- 0.0
   for t from 1 to T
       sumup <- sumup + f(T)
   end
   return sumup
end
```

Infinite Sums?

How can an infinite sum be worth less than infinite cash?

- Because each future C is worth *a lot* less than the preceding C.
- In the graph on the next page, the PV of each cash flow is the bar's area.
- Soon, terms add almost nothing.

Graph: Perpetuity

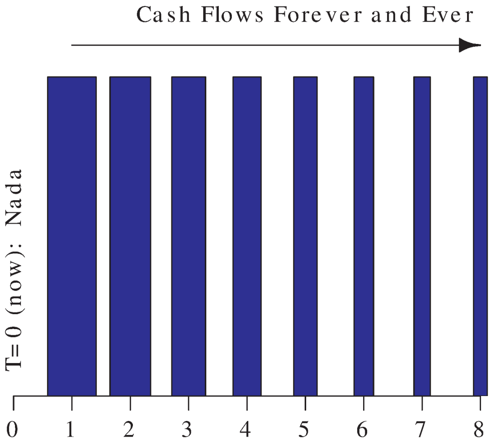

Figure 1: Cash Flows of P in Today's Value

Value of Perpetuity I

What is the value of an unbreakable promise to receive \$10 forever, beginning next year, if the interest rate is 5% per year?

Value of Perpetuity II

What is the value of an unbreakable promise to receive \$10 forever, beginning **this** year, if the interest rate is 5% per year?

Perpetuity Formula Mod

What is the perpetuity formula if the first cash flow starts today rather than tomorrow?

Nerd: Time Consistency

- Assume an interest rate of 10%.
- A perpetuity today with $1 forever is worth $ $1/0.1= $10 $.
- A perpetuity tomorrow with $1 forever will be worth $1/0.1= $10 tomorrow.
- Today's perpetuity gives you $1 extra next period, and leaves you with a then $10 perpetuity. At 10%, they are worth $1/(1+10%) and and $10/(1+10%), respectively. The latter is next year's perpetuity.

Growing Perpetuities

A growing perpetuity pays

- C next year
- then $C \cdot (1 + g)$ the following year,
- then $C \cdot (1 + g)^2$ the following year,
- then ...

Growing perpetuities generalize simple perpetuities ($g = 0$).

Growing Perpetuity Table of Cash Flow and Present Values, g=10%

Time	Cash Flow	Is Worth Today
0	\$0	\$0
1	\$100	\$100
2	\$100*1.1	\$110
3	\$100*1.1^2	\$121
4	\$100*1.1^3	\$133
5	\$100*1.1^4	\$146
...	...	...
t	\$100*1.1^t	...
...	...	...

Growing Perpetuities Formula

The PV of a growing perpetuity is

$$PV(C_1, g, r) = \sum_{t=1}^{\infty} \frac{C_1 \cdot (1+g)^{t-1}}{(1+r)^t} \ .$$

The real beauty is the shortcut formula,

$$PV(C_1, g, r) = \frac{C_1}{r-g} \ .$$

Growing Perpetuities Footnotes

You must memorize the shortcut formula, and know what it means!

- The growth term g acts like a reduction in the interest rate r.
- The time subscript for the payment matters now, because $C_1 \neq C_2 \neq C_t$.

(NFL) Booth Review

Check the growing perpetuity formula by hand!

Infinite Sums?

How can an infinite sum be worth less than infinite cash?

- Because the growth g is not too fast.
- Each rectangle is smaller than the preceding one, i.e., each PV is smaller than the preceding one.

What if $g \geq r$?

- The formula then makes no sense.

Graph: Growing Perpetuity

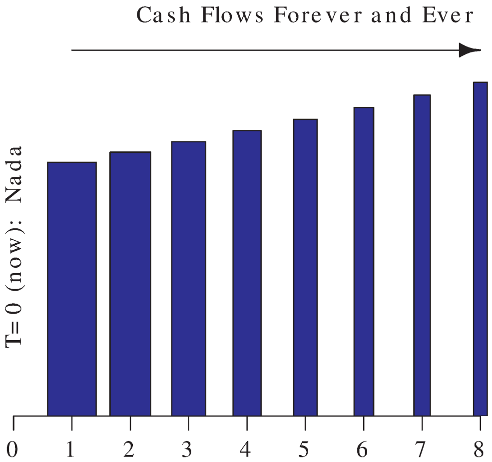

Figure 2: Cash Flows of GP in Today's Value

Value of Eternal Guarantee

What is the value of a guarantee to receive \$10 next year, growing by 2%/year (just the inflation rate) forever, if the interest rate is 6%/year?

Growing Formula Mod

What is the value of a firm that just paid \$10 **this** year, growing by 2%/year forever, if the interest rate is 5%/year?

Example PV Calc I

What is the formula for the value of a firm which will only grow at the inflation rate, and which will have \$1 million of earnings next year?

Example PV Calc II

In 10 years, a firm will have annual cash flows of $100 million.

Thereafter, its cash flows will grow at the inflation rate of 3%.

If the applicable interest rate is 8%, estimate its value if you will sell the firm in 10 years?

What would this “terminal value” be worth today?

Pro Forma TVE

Terminal Value Estimates are the most common use of the formula:

- guestimate the PV of the firm after an arbitrary T years in the future.
- The inflation rate is often the common long-run growth rate, g.
- A typical T in a "pro-forma" would be 5-10 years.

Gordon Dividend Growth Model

What should be the share price of a firm that

- pays dividends of \$1/year,
- whose dividends grow by 4% every year, and
- which will continue to do so forever,
- if its cost of capital (CoC) is 12%/year?

GDGM for CFAs

CFA Exam: Using D for C gives you the GDGM.

$$P = \frac{D}{r - g} \, .$$

Ergo $D/P = r - g$.

GDGM for Real

Don't trust the GDGM

- Firms can shift dividends!
- What a firm does not pay out in dividends today will make more hey (dividends) tomorrow.
- it should not matter if the firm cancels its $1 dividends this year in order to pay out an extra $1.05 next year.

GDGM Improvements?

- An improvement uses the **plowback ratio**:
  - it takes into account that reinvested cash should pay more dividends in the future,
  - but it's still just lipstick on a pig.

- A better valuation formula could use earnings instead of dividends,
  - because earnings are more difficult to shift around.

GDGM Implied Cost of Capital (ICC)

What is the CoC for a firm that

- pays a dividend yield (D/P) of 5%/year today,
- if its dividends are expected to grow at a rate of 3%/year forever?

GDGM ICC Formula

An *Implied Cost of Capital* (*ICC*) is the expected RoR embedded in the stock price today.

- GDGM is sometimes used to estimate an implied cost of capital, ICC,
- via the inverted formula $r = D/P + g$.
- A higher P today implies a lower implied CoC at which the firms can obtain capital from investors.

S&P500 ICC

Using Goyal-Welch Macro Data:

If stocks will grow roughly at the GDP growth rate of 4-5% per year,
what should investors reasonably expect about future RoR implied by a P/E ratio of 24?

- 2017 S&P500 $P/E \approx 24$.

Quick Calc: Value of Firm

Our firm has earned \$100,000 this year.

It has stopped growing in *real* terms.

The current interest rate is 6%/year.

The inflation rate is 2%/year.

What is the value of our firm?

- What is it over-the-envelope ?
- What is it exactly?
- What is the first cash flow?

Growth Rate of Google

In April 2020, Alphabet (Google)'s share price was about \$1,260.

Trailing twelve months (TTM) EPS was \$50.

Therefore, Google's P/E Ratio was about 25.

Google's CoC was about 8%/y.

What does the market believe G's as-if-eternal earnings growth rate will be?

Metaphysics

Are perpetuities meaningful?

- How long will firms last?
- How long will Google last?
- What firms or institutions have survived from the Roman empire?