Annuities

(Welch, Chapter 03-B)

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Annuities

An **annuity** is a financial instrument that pays C dollars for a given T number of periods.

Specific Sample Questions

- How are loan payments computed?
- What is the monthly payment on a 5% 30-year fixed rate mortgage?

Key Annuities Formula

The PV formula for an annuity is

$$PV = \sum_{t=1}^{T} \frac{C}{(1+r)^t} = \left(\frac{C}{r}\right) \cdot \left[1 - \frac{1}{(1+r)^T}\right]$$

- Make sure you know when the first cash flow begins: **tomorrow** $t = 1$, not today $t = 0$!
- **Memorize this formula!**

Annuities Formula

Another way to write the formula:

$$PV(C_1, r, T) = \left(\frac{C_1}{r}\right) - \left[\frac{1}{(1+r)^T}\right] \cdot \left(\frac{C_1}{r}\right)$$

Conceptual Note:

- An annuity is one perpetuity today minus a discounted future perpetuity.

Value of a 1-Year Annuity

Assume C=\$10, r=5%, T=1.

Simple:

$$PV = \frac{\$10}{1+5\%} = \$9.52.$$

Via Formula:

$$PV = \frac{\$10}{5\%} - \frac{1}{1+5\%} \cdot \frac{\$10}{5\%} = \$9.52.$$

Value of a 2-Year Annuity

Assume C=\$10, r=5%, T=2.

Simple:

$$PV = \frac{\$10}{1+5\%} + \frac{\$10}{(1+5\%)^2} = \$18.59.$$

Via Formula:

$$PV = \frac{\$10}{5\%} - \frac{1}{(1+5\%)^2} \cdot \frac{\$10}{5\%} = \$18.59.$$

Value of a 30-Year Annuity

Assume C=\$10, r=5%, T=30.

Simple:

(You do this, not me)

Via Formula:

$$PV = \frac{\$10}{5\%} - \frac{1}{(1+5\%)^{30}} \cdot \frac{\$10}{5\%} = \$153.70.$$

Example: Mortgage Loan

How mortgage payments are calculated:

- A 30-year mortgage is an annuity with 360 monthly payments, starting next month.
- The monthly interest rate is the quoted rate divided by 12.
- Example: The monthly interest rate on a 9% mortgage is

$$r_{\text{monthly}} = 0.09/12 = 0.0075 \text{ per month}$$

(You could also call this 9%, compounded monthly.)

Monthly Mortgage Payment

To buy a house, say you need to take out a $1,200,000 fixed-rate mortgage with 30 years to maturity, 360 equal monthly payments, and a quoted interest rate of 9%.

What will be your monthly mortgage payment?

Principal and Interest

Of the first month's payment, how much is interest and how much is principal repayment?
Important!

- Interest is tax deductible, principal repayment is not.
- If you want to pay off the mortgage, you only pay the remaining balance.

Principal and Interest, Month 1

- The monthly interest rate is 0.75%. The interest payment is

$$0.0075 \cdot \$1,200,000 = \$9,000.00 \ .$$

- The remaining $\$9,655.47 - \$9,000 \approx \$655.47$ pays off (some of the remaining) principal on the loan.
- After the 1st payment, the loan balance is

$$\$1,200,000 - \$655.55 \approx \$1,199,344.53 \ .$$

Principal and Interest, Month 2

- Interest charged in month 2 is

$$\$1,199,344.53 \cdot 0.0075 \approx \$8,995.08 \ .$$

- \$8,995.08 of month 2's payments is to interest, the remaining is to principal:

$$\$9,655.47 - \$8,995.08 \approx \$660.39 \ .$$

- The remaining balance on the mortgage after your second mortgage payment is

$$\$1,199,344.53 - \$660.387 \approx \$1,198,684.14 \ .$$

Principal and Interest, Month 3

- Interest Payment

$$\approx \$1,198,684.14 \cdot 0.0075 \approx \$8,990.13 \ .$$

- Principal Repayment

$$\approx \$9,655.47 - \$8,990.13 \approx \$665.34 \ .$$

- Remaining balance

$$\approx \$1,198,684.14 - \$665.34 \approx \$1,198,018.80 \ .$$

The Car Dealer Loan

An example drawn from an actual automobile loan agreement: The advertisement claimed,

12 month car loans. Only 9%!

- In fine-print: For a 12-month \$10,000 loan at 9%, you owe \$10,900. Your 12 monthly payments will be $\$10,900/12 \approx \$908.33$ per month.

OK?

A 9% Bank Loan

If you took out a \$10,000 loan from the bank at a true interest rate of 9% (8.649% compounded monthly), how much would you have to pay each month?

Automobile Loan I

Whence the payment difference?

Automobile Loan II

What is the car dealer's true IRR?

Painful?

Learning the principles and working details is important if you do not want to be taken advantage of.

- Each month, you have paid off part of the principal, thereby borrowing less later in the year.
- The interest rate of the dealer assumes that you borrow all \$10,000 for the whole year.
- Watch out how you are being charged! Finance can be used to trick unsuspecting victims.

Level-Coupon Bonds

Bonds are long-term loans, typically taken out by large institutions and typically resellable.

- Most bonds are *coupon bonds*; i.e., they make interim "coupon" payments.
- Most bonds are *level-coupon bonds*.
  - The coupon payments are all the same.
- Most corporate bonds are *x% semi-annual level coupon bond*.
  - They pay the same coupon twice a year.

Semi-Annual Level Coupon Quote

- Take the principal (often \$1,000 for corporate bonds), multiply it by x% to obtain the annual coupon payment, divide it by two, and this is the coupon that is paid every six months.
- **Example**: An 8% semi-annual level coupon bond pays \$40 every six months on \$1,000 in principal. At maturity, it pays \$1,040.
  - **The 8% is not the implicit interest rate of the bond!**
  - 8% is just a standard way to tell you the coupon flows.

Coupon Bond Payments

Describe the payments to a 5% semi-annual level coupon bond, $100 million, due in 30 months.

Zero Bond

A zero bond has no interim payments.

How do you earn interest on a bond that gives you no interest payments?

Coupon vs Interest Rate

Is the coupon rate of a bond equal to the interest rate?

Interest Rate on Coupon Bonds

What is the implied interest rate on a Walmart 3.5% semi-annual coupon bond?

Retirement Instruments

- You can purchase quasi-perpetuities for *your* life.
- You can purchase term-life insurance.
- You can purchase annuities for retirement purposes.
- The retirement annuities industry is "only" about $3 trillion in size!
- One day, you will care!

Retirement Annuity Example

An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years and sells for \$806,070.

What is the implied interest rate (here called an IRR—more soon) that this insurance company is offering you?

Retirement Annuity Example

An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years, growing at an "inflation-compensator" rate of 3%, and sells for \$806,070.

What is the implied interest rate?

DIY Retirement

The prevailing interest rate is 10%/year.

If you put aside $1,000,000 to cover 18 years of expenses, how much could you draw down each year?

DIY Per Year

The prevailing interest rate is 10%/year.

If you want to draw \$100,000 each year to cover 18 years of expenses, how much would you have to set aside?

Pro Forma Terminal Value

What fraction of a perpetuity's value comes from the first t years? I.e., How reasonable an approximation is a perpetuity for an annuity?

$$PV(P) - PV(A) = 1 - \frac{1}{(1+r)^t} \ .$$

- This fraction is larger if r and t are bigger.
  - For $r = 5\%$, 62% for 20 years, 77% for 30 years.
  - For $r = 10\%$, 85% for 20 years, 94% for 30 years.
- For high interest-rate (risky) cash flows, predicting 10-20 years out is mostly the same.

Comparing Lease Options

Assume the interest rate is $r = 20\%$. You need to lease a building. The landlord gives you two choices, payments due at year start:

A. A two-year lease at \$12,000/y, plus a one-time extra upfront payment of \$8,000.

B. A three-year lease at \$15,000/y.

Such lease options are commonly available, e.g., for cars see the Bankrate.com Lease Calculator.

Which Lease is Better?

- A costs $32 total for 2 years
- B costs $45 but is for 3 years.
- But A takes more money up-front.
- But B is cheaper per year.

How Can You Compare Leases?

What is the “equivalent annual rent” of Lease A?

Comparing Leases — Assumptions?

- What if you need to use the building for exactly 2 years (no sublets!)?
- What if you need to use the building for exactly 3 years?
- What if a 3-y old building is worse than a 2-y old building?

APPENDIX

Chapter 2 also contains

- Proof of Formulas.
- The formula for a *growing annuity*, rarely needed.
  - Beloved only by sadistic finance professors for exam questions.