Time-Varying Rates of Return, Bonds, Yield Curves

(Welch, Chapter 05)

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Maintained Assumptions

Perfect Markets

1. No differences in opinion.
2. No taxes.
3. No transaction costs.
4. No big sellers/buyers—infinitely many clones that can buy or sell.

Perfect Certainty

BUT NO LONGER Equal Returns Per Period

Time-Varying Preferences

Oranges cost more in the winter than in the summer, because they are scarcer.

Maybe investors like bonds more if they come due sooner? Or bonds that come due when they retire?

Generalization of Constant RoRs

All earlier formulas hold!

- The only difference is that $(1 + r_{0,t}) \neq (1 + r)^t$.
- The main complication is that we now need many subscripts—one for each period.

Rates of Return (T=3)

$$(1 + r_{0,3}) = (1 + r_{0,1}) \cdot (1 + r_{1,2}) \cdot (1 + r_{2,3}) \ .$$

$$(1 + r_{0,3}) = (1 + r_1) \cdot (1 + r_2) \cdot (1 + r_3) \ .$$

- Recall that r_j is an abbrev for $r_{j-1,j}$.

NPV (T=3)

$$C_0 + \frac{C_1}{(1+r_{0,1})} + \frac{C_2}{(1+r_{0,2})} + \frac{C_3}{(1+r_{0,3})}.$$

$$= C_0 + \frac{C_1}{(1+r_{0,1})} + \frac{C_2}{(1+r_{0,1})\cdot(1+r_{1,2})} + \frac{\quad}{(1+r_{0,1})\cdot(1}$$

Time-Varying Rates of Returns

More formal,

$$(1 + r_{t,t+i}) = (1 + r_{t,t+1})$$

$$\cdot(1 + r_{t+1,t+2}) \cdots (1 + r_{t+i-1,t+i})$$

$$= (1 + r_{t+1}) \cdot (1 + r_{t+2}) \cdots (1 + r_{t+i}) = \prod_{j=t+1}^{t+i} (1 + r_j) \ .$$

- Recall that r_j is an abbrev for $r_{j-1,j}$.

Present Value

$$PV = \sum_{t=1}^{\infty}\left[\frac{\mathrm{CF}_t}{(1+r_{0,t})}\right] = \sum_{t=1}^{\infty}\left[\frac{\mathrm{CF}_t}{\prod_{j=1}^{t}(1+r_j)}\right].$$

. . . In Non-Math Language

Here is a computer program that executes this formula.

It relies on two functions:

- CF(t) is cashflow at time time t
- Interest rate from $t-1$ to t is $r(t-1,t)$.

Computer Program

```
df <- 1.0
PV <- 0.0
for time t=0 to infinity do
    df <- df/( 1+r(t-1,t) )
    PV <- PV + CF(t) * df
return PV
```

Inflation and Real Rates

see c05-inflation.pdf.

Treasury Fixed Income

Warning: By necessity, this is the most algebra-heavy subject in finance. It is all about interest rates.

It is also the most applied and practical material in the book!

US Treasuries Background I

US Treasuries are the most important financial security in the world.

Outstanding amounts in 2019:

- US Treasuries , ≈\$17 trillion
- Mortgage Bonds , ≈\$10 trillion
- Corporate Bonds , ≈\$9 trillion
- Muni Bonds , ≈ \$4 trillion

US Treasuries Background II

Annual trading is ≈\$100-\$150 trillion.

- Turnover = 5-10 Times!

Bond Names:

- Bills (–0.99yr)
- Notes (1yr–10yr)
- Bonds (10yr–).

US Treasuries Background III

This market is close to “perfect”:

- Extremely low transaction costs (for traders).
- Few opinion differences (inside information).
- Deep market—many buyers and sellers.
- Income taxes depend on owner.

In addition, there is (almost) no uncertainty about repayment.

- PS: a market could still be perfect, even if payoffs are uncertain.

The Yield Curve (YC)

The yield curve is the plot of annualized yields (Y-axis) against time-to-maturity (X-axis).

(Zero-coupon = Strip) US Treasuries are the *simplest* financial instrument in the world.

Graph: YC Dec 2015

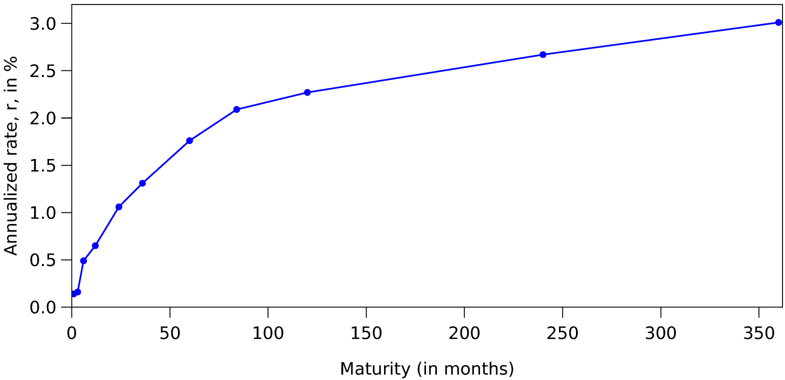

Figure 1: Yield Curve, Dec 2015

Treasury YC Slopes

Can the Treasury yield curve be flat?

Can it slope up?

Can it slope down?

Graph: YC Jan 2007

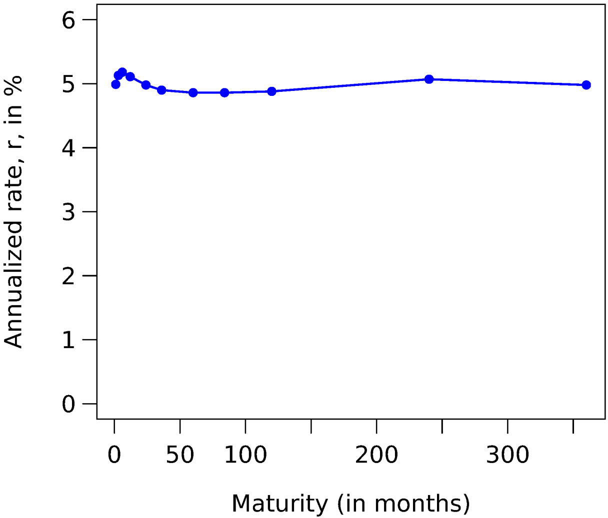

Figure 2: Yield Curve, Jan 2007

Graph: YC Dec 1980

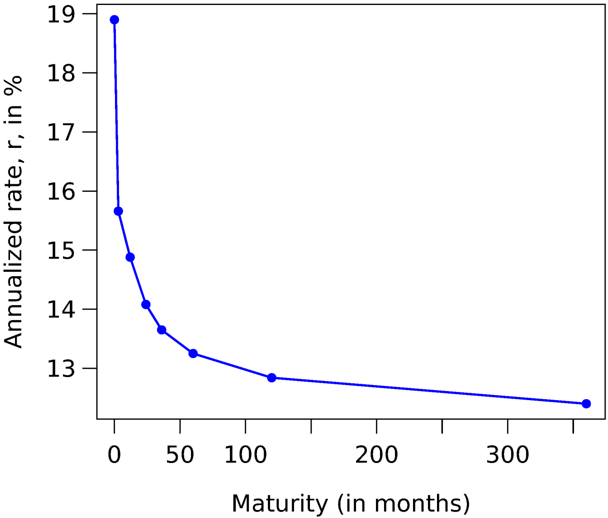

Figure 3: Yield Curve, Dec 1980

“Term Structure”

A **yield curve** is a fundamental tool of finance.

- It always graphs *annualized* rates.
- It measures differences in the costs of capital for (risk-free) projects with different horizons.
- The most important yield curve is the US Treasuries Yield Curve
  - Default, unless otherwise specified.

“Term Structure”

In the real world, many variations on the yield curve are in use.

Treasury, TIPS, Muni

Germany

Corporate Bonds.

Many Others

Better Deal?

In a PCM, is a 3-year T-note with a higher interest rate a better deal than a 3-month T-bill with a lower interest rate?

Common YC Slope

What is the most common yield curve shape?

Meaning of YC Slope

What does an upward sloping or downward sloping yield curve mean for the economy (not for an investor)?

Fed Control?

Does the Fed control the (Treasury) yield curve?

Spot and Forward Rates

Spot interest rate: a currently prevailing interest rate for an investment starting today.

Forward (interest) rate: an interest rate that will begin with a cash investment in the future.

- This is the opposite of a spot rate.

Like all other interest rates, spot and forward rates are usually quoted in annualized terms.

Annualized Spot Rate

What is the annualized spot rate on a 1-month US T-bill today?

Annualized Spot Rate

What is the annualized spot rate on a 30-year US T-bond today?

Future Interest Rates?

What does the yield curve today imply about future interest rates?

Can you lock in future interest rates today?

Double-Subscript Painful Notation

An annualized interest rate over 15 years is denoted as: $r_{\overline{15}}$

- This contrasts with our notation for the 15-year non-annualized holding interest rate ($r_{0,15}$).

$$(1+r_{\overline{15}})^{15} \equiv (1+r_{0,15}) \equiv (1+r_{0,1})\cdot(1+r_{1,2})\cdot\ldots\cdot(1+r_{14,15}) \ .$$

$$\iff (1+r_{\overline{15}}) \equiv (1+r_{0,15})^{1/15} \ .$$

General Annualized Rates

$$(1 + r_{\bar{t}})^t \equiv (1 + r_{0,t}) \ .$$

$$\iff (1 + r_{\bar{t}}) \equiv (1 + r_{0,t})^{1/t} \ .$$

Notation Non-Generality

- There is no standard notation for annualized.
- Overbar is our notation, reminiscent of average.
- Some use R but mean $1+r$. Others mean r by R. Some use rf. etc.
- Ask!

Out Example: $r_{0,5} = 27.63\% \iff r_{\overline{5}} = 5\%$.

Notation Summary

$$(1 + r_{0,1}) = (1 + r_{\overline{1}})^1 = (1 + r_{0,1}) \ .$$

$$(1 + r_{0,2}) = (1 + r_{\overline{2}})^2 = (1 + r_{0,1}) \cdot (1 + r_{1,2}) \ .$$

$$(1+r_{0,3}) = (1+r_{\overline{3}})^3 = (1+r_{0,1}) \cdot (1+r_{1,2}) \cdot (1+r_{2,3}) \ .$$

- Now: year 0.
- The interest rate from year 1 to year 2 is the 1-year *forward* rate.
- In a world of certainty, the forward rate will be the future spot rate: We know it!

Approximate Rates

An annualized rate is more like an average.

A holding rate is more like a sum.

Chained Bonds

A 1-year bond has a RoR of 5%.

When it will come due, you will be able to purchase another 1-year bond that will have an (annual) RoR of 10%.

When this second bond will come due, you will be able to purchase another 1-year bond that will have an (annual) RoR of 15%.

Worksheet

What are the three total *holding* RoRs?
What are the three *annualized* RoRs?

(Calculator VERBOTEN. Use your intuition.)

Table of Rates of Return

T (End)	Spot + Forward	Holding	Annualized
1			
2			
3			

Three Holding Rates

What are the three *holding* rates exactly?

Three Annualized Rates

What are the three annualized interest rates exactly?

Next Example

The 1-year bond annualized RoR is 5%/y.
The 2-year bond annualized RoR is 10%/y.
The 3-year bond annualized RoR is 15%/y.

- **First w/o a calculator, then with.**

Holding Rates of Return

- **First w/o a calculator, then with.**

What are the three holding RoRs?

What are the three spot and future RoRs?

Fill In

T (End)	Spot + Forward	Holding	Annualized
1			
2			
3			

Assess Magnitudes First

- Use over-the-envelope intuition for magnitudes.
- Because the annualized yield is an average of spot/forward rates, the forward rates rises/declines faster than the yield curve.
- Example: if $r_{\overline{1}} = 5\%$ and $r_{\overline{2}} = 6\%$, then $r_{1,2} > 6\%$, because 5% and $r_{1,2}$ "geo-averaged" must come to 6%.
- By this argument, $r_{1,2}$ should be about 7%.

Equivalence of Curves

The following contain the same information:

- full set of annualized rates (yield curve),
- full set of spot and forward rates,
- full set of holding rates (0 to y).

Each can be translated into the others.

YC Summary

The *Yield Curve* (YC) is the *Term Structure of Interest Rates*, with the curve plotting

- the annualized interest rate on the y-axis
- against the time of the payment on the x-axis.

Nerd: Treasury Strips

Although we pretend that YC are based on true x-year *strips* (interest rates), usually they are from interest rates from x-year coupon bonds.

The duration for such bonds is shorter than their maturity. Usually, the yield difference is small.

Strips are the real thing: zero-coupon bonds.

Unless you are a bond trader, you can probably ignore this difference.

Upward-Sloping YCs

What does an upward-sloping yield curve mean for an investor?

- 4A, Higher future inflation? (not usually)
- 4B, Higher future interest rates? (not usually)
- 4C, Bargains? (not usually)

Upward-Sloping YC II

- 4D, Risk Compensation? (most likely, yes)
  - In the real world, you have a choice:
    - Lock in future interest rates (gives you what we calculated).
    - There is very little transaction cost to do this (financial markets are close-to-perfect)
    - Take your chances: future actual interest rates may be higher/lower than the interest rates you could lock in today.
  - "risk premium": risk is higher for longer-term investments
    - e.g.: if the firm can go bankrupt or inflation may erode the value of the repayment
    - Implies an upward-sloping yield curve.

Interest Rate Sensitivity

What happens to the value of a bond (a loan) that you already own when interest rates increase?

Does the loan length matter?

Example: 30-Year Bond

A 30 year bond that promises 8% interest rate costs ($\$100/1.08^{30} \approx$) \$9.94 for each \$100 promise in payment.

If the interest rate increases by 10 basis points, the price changes to \$9.67.

The holding RoR is

$$\$9.67/\$9.94 - 1 \approx -2.74\% \ .$$

For each \$100 in investment, you would have just lost \$2.74!

Example: Shorter Bonds

For a 1-year bond, the same calculation

$$p_0 = \$100/1.08 \approx \$92.5926 \ .$$

$$p_1 = \$100/1.081 \approx \$92.507 \ .$$

$$r = p_1/p_0 - 1 \approx -0.09\% \ .$$

For a 1-day bond, the calculation

$$p_0 = \$100/1.08^{1/365} \approx \$99.979 \ .$$

$$p_1 = \$100/1.081^{1/365} \approx \$99.9787 \ .$$

Relative Sensitivity

The interest rate sensitivity of a 30-year bond is higher than that of a 1-year (or 1-day bond).

Which is Riskier?

If 10bp economy-wide interest rate changes are equally likely for 30-year as 1-day rates, then 30-year bonds are riskier investments.

In the real world, short-rates changes of 10bp are more common for short (1-year) economy-wide rates than for long (30-year) ones.

. . . but not so common as to negate the fact that the 30-year is riskier than the 1-year.

Risk or Interest Rate Expectation?

If we allow for uncertainty, long-term bond investors usually get more yield for 2 reasons:

- because of higher expected RoRs in the future (e.g. due higher future inflation rates),
- and/or because they are earning a "risk premium" (to be discussed soon).

Empirical Evidence: primarily risk premium.

YC Corporate Lessons I

A project of x-years is not simply the same as investing in x consecutive 1-year projects.

- Different animals.
- They can require different costs of capital.

The fact that longer-term projects may have to offer higher RoRs (could but) need not be due to higher risk.

- Even default-free Treasury bonds do so.

YC Corporate Lessons II

Of course, long-term projects are also often riskier. (They default more often.)

This also contributes to why long-term projects have to offer higher RoRs.

Project Duration I

A project that pays \$200 in 1 year and \$100 in 4 years has a maturity of 4 years;

Another project that pays \$300 in 4 years has the same maturity.

However, the first project is clearly shorter-term.

How do we measure: When does the average cash flow arrive?

Project Duration II

$$\frac{1 \times \$200 + 2 \times \$0 + 3 \times \$0 + 4 \times \$100}{\$200 + \$0 + \$0 + \$100} = 2\ .$$

- Numerator is cash flow times timing.
- Denominator is sum of cash flows.

Macauley Duration

Macauley duration uses the PV and not raw cash flows. If $r = 10\%$,

$$\frac{1 \times \$200/1.1 + 4 \times \$100/1.1^4}{\$200/1.1 + \$100/1.1^4} = 1.82 \; .$$

Macauley duration tilts more towards the front (i.e., it is smaller than plain duration).

App: Locked Forwards

Locking In Forward Rates:

- Given the current yield curve, you can lock in the future interest rate today.
- That is, you can eliminate all uncertainty about what interest rate that you will have to pay (or that you can earn).
- **Example**: Buy and short Treasuries to lock in a 1-year Treasury rate for $1 million beginning in year 3 and lasting until year 4.

App: Future vs Forward I

Future Interest Rates vs Forward Rates:
In the real world, future interest rates can be different from forward rates.

- If you lock in a 10-year-ahead 1-year savings interest rate today, on average you would have earned a higher RoR than you would have if you had purchased 1-year savings bonds in the open market.

App: Future vs Forward II

If you are dealing with bonds, you therefore may need more notation.

- You now will have a future 1-year spot rate in 2030 (say $r_{2030,2031}$), and a 1-year forward rate that you can lock in today (say $f_{\text{Now},2030,2031}$, which is the 1-year forward rate locked in today).
- Tomorrow's locked in forward rate would be $f_{\text{tomorrow},2030,2031}$, and so on. Yikes.

App: Continuous Compounding

If interest is paid not once per year, but every second, this is the continuously compounded interest rate.

CC is used heavily in option pricing.

Example: 10% once; 5% twice= 10.25%; 1% ten times= 10.46%; 0.1% 100 times= 10.51%; eventually $e^0.1 - 1 \approx 10.5171\%$.