

# Annuities

(Welch, Chapter 03-B)

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Thu Jun 23 23:24:47 2022

# Annuities

- ▶ An **annuity** is a financial instrument that pays  $C$  dollars for a given  $T$  number of periods.
- ▶ Most loans etc. are annuities. Thus, it is important that you be able to work with them.

# Specific Sample Questions

- ▶ How are loan payments computed?
- ▶ What is the monthly payment on a 5% 30-year fixed rate mortgage?

# Key Annuities Formula

The PV formula for an annuity is

$$PV = \sum_{t=1}^T \frac{C}{(1+r)^t} = \left(\frac{C}{r}\right) \cdot \left[1 - \frac{1}{(1+r)^T}\right]$$

- ▶ Make sure you know when the first cash flow begins: **tomorrow**  $t = 1$ , not today  $t = 0$ !
- ▶ **Memorize this formula!** *Really!!*

# Annuities Formula

Another way to write the formula:

$$PV(C_1, r, T) = \left( \frac{C_1}{r} \right) - \left[ \frac{1}{(1+r)^T} \right] \cdot \left( \frac{C_1}{r} \right)$$

Conceptual Note:

- ▶ An annuity is one perpetuity today minus a discounted future (time T) perpetuity.

# Value of a 1-Year Annuity

Assume  $C=\$10$ ,  $r=5\%$ ,  $T=1$ .

▶ Simple:

$$PV = \frac{\$10}{1 + 5\%} = \$9.52.$$

▶ Via Formula:

$$PV = \frac{\$10}{5\%} - \frac{1}{1 + 5\%} \cdot \frac{\$10}{5\%} = \$9.52.$$

# Value of a 2-Year Annuity

Assume  $C=\$10$ ,  $r=5\%$ ,  $T=2$ .

▶ Simple:

$$PV = \frac{\$10}{1 + 5\%} + \frac{\$10}{(1 + 5\%)^2} = \$18.59.$$

▶ Via Formula:

$$PV = \frac{\$10}{5\%} - \frac{1}{(1 + 5\%)^2} \cdot \frac{\$10}{5\%} = \$18.59.$$

# Value of a 30-Year Annuity

Assume  $C=\$10$ ,  $r=5\%$ ,  $T=30$ .

▶ Simple:

(You do this, not me — I am too lazy.)

▶ Via Formula:

$$PV = \frac{\$10}{5\%} - \frac{1}{(1 + 5\%)^{30}} \cdot \frac{\$10}{5\%} = \$153.70.$$



## Example: Mortgage Loan

How mortgage payments are calculated:

- ▶ A 30-year mortgage is an annuity with 360 monthly payments, starting next month.
- ▶ The monthly interest rate is the quoted rate div by 12.
- ▶ Example: Monthly interest rate on a 9% mortgage is

$$r_{\text{monthly}} = 0.09/12 = 0.0075 \text{ per month}$$

(You could also call this 9%, compounded monthly.)

# Monthly Mortgage Payment

- ▶ Say you need to take out a \$1,200,000 fixed-rate mortgage with 30 years to maturity, 360 equal monthly payments, and a quoted interest rate of 9%.
- ▶ What will be your monthly mortgage payment?

# Principal and Interest

- ▶ Of the first month's payment, how much is interest and how much is principal repayment?
- ▶ Why is this important?
  - ▶ Interest is tax deductible, principal repayment is not.
  - ▶ If you want to pay off the mortgage, you only pay the remaining balance.

# Principal and Interest, Month 1

- ▶ The monthly interest rate is 0.75%. The interest payment is

$$0.0075 \cdot \$1,200,000 = \$9,000.00 .$$

- ▶ The remaining  $\$9,655.47 - \$9,000 \approx \$655.47$  pays off (some of the remaining) principal on the loan.
- ▶ After the 1st payment, the loan balance is

$$\$1,200,000 - \$655.55 \approx \$1,199,344.53 .$$

# Principal and Interest, Month 2

- ▶ Interest charged in month 2 is

$$\$1,199,344.53 \cdot 0.0075 \approx \$8,995.08 .$$

- ▶ \$8,995.08 of month 2's payments is to interest, the remaining is to principal:

$$\$9,655.47 - \$8,995.08 \approx \$660.39 .$$

- ▶ The remaining balance on the mortgage after your second mortgage payment is

$$\$1,199,344.53 - \$660.387 \approx \$1,198,684.14 .$$

# Principal and Interest, Month 3

- ▶ Interest Payment

$$\approx \$1,198,684.14 \cdot 0.0075 \approx \$8,990.13 .$$

- ▶ Principal Repayment

$$\approx \$9,655.47 - \$8,990.13 \approx \$665.34 .$$

- ▶ Remaining balance

$$\approx \$1,198,684.14 - \$665.34 \approx \$1,198,018.80 .$$

# Advertised Special Loan

- ▶ **12 month car loans. Only 9%!**

- ▶ In fine-print: For a 12-month \$10,000 loan at 9%, you owe \$10,900. Your 12 monthly payments will be  $\$10,900/12 \approx \$908.33$  per month.
- ▶ Such quotes are common in some countries without laws prohibiting them.

OK?



## A 9% Bank Loan

If you took out a \$10,000 loan from a legitimate bank at a true interest rate of 9% (8.649% compounded monthly), how much would you have to pay each month?

# Automobile Loan I

Whence the payment difference?

# Automobile Loan II

What is the car dealer's true IRR?

# Painful?

Learning the principles and working details is important if you do not want to be taken advantage of (the “mark”).

- ▶ Each month, you have paid off part of the principal, thereby borrowing less later in the year.
- ▶ The interest rate of the dealer assumes that you borrow all \$10,000 for the whole year.
- ▶ Watch out how you are being charged! Finance can be used to trick unsuspecting victims.

# Level-Coupon Bonds

Bonds are long-term loans, typically taken out by large institutions and typically resellable.

- ▶ Most bonds are *coupon bonds*; i.e., they make interim “coupon” payments.
- ▶ Most bonds are *level-coupon bonds*.
  - ▶ The coupon payments are all the same.
- ▶ Most corporate bonds are  *$x\%$  semi-annual level coupon bond*.
  - ▶ They pay the same coupon twice a year.

# Semi-Annual Level Coupon Quote

- ▶ Take the principal (often \$1,000 for corporate bonds), multiply it by  $x\%$  to obtain the annual coupon payment, divide it by two, and this is the coupon that is paid every six months.
- ▶ **Example:** An 8% semi-annual level coupon bond pays \$40 every six months on \$1,000 in principal. At maturity, it pays \$1,040.
  - ▶ **The 8% is not the implicit interest rate of the bond!**
  - ▶ 8% is just a standard way to tell you the coupon flows.

# Coupon Bond Payments

Sep 2020. American Honda Finance. 1.0% Medium-Term Notes due Sep 2025. A3 Rating. \$750 million.

Describe the exact interest and principal payment pattern.

# Zero Bond

A zero bond has no interim payments.

How do you earn interest on a bond that gives you no interest payments?



# Coupon vs Interest Rate

Is the coupon rate of a bond equal to the interest rate?

# Interest Rate on Coupon Bonds

What is the implied interest rate on a Walmart 3.5% semi-annual coupon bond?

# Retirement Instruments

- ▶ You can purchase quasi-perpetuities for *your* life.
- ▶ You can purchase term-life insurance.
- ▶ You can purchase annuities for retirement purposes.
- ▶ The retirement annuities industry is “only” about \$3 trillion in size!
- ▶ One day, you will care!

# Retirement Annuity Example

An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years and sells for \$806,070.

What is the implied interest rate (here called an IRR—more soon) that this insurance company is offering you?

# Retirement Annuity Example

An insurance company offers a retirement annuity that pays \$100,000 per year for 15 years, growing at an “inflation-compensator” rate of 3%, and sells for \$806,070. What is the implied interest rate?

# DIY Retirement

The prevailing interest rate is 10%/year.

If you put aside \$1,000,000 to cover 18 years of expenses, how much could you draw down each year?

## DIY Per Year

The prevailing interest rate is 10%/year.

If you want to draw \$100,000 each year to cover 18 years of expenses, how much would you have to set aside?

# Pro Forma Terminal Value

- ▶ What fraction of a perpetuity's value comes from the first  $t$  years?
- ▶ How reasonable an approximation is a perpetuity for an annuity?
- ▶ If you incorrectly used a perpetuity (quicker to calculate), is this a forgivable sin?



$$PV(P) - PV(A) = 1 - \frac{1}{(1+r)^t} .$$

- ▶ This fraction is larger if  $r$  and  $t$  are bigger.
  - ▶ For  $r = 5\%$ , 62% for 20 years, 77% for 30 years.
  - ▶ For  $r = 10\%$ , 85% for 20 years, 94% for 30 years.
- ▶ For high interest-rate (risky) cash flows, predicting 10-20 years out is mostly the same.

# Comparing Lease Options

Assume the interest rate is  $r = 20\%$ . You need to lease a building. The landlord gives you two choices, payments due at year start:

1. A two-year lease at \$12,000/y, plus a one-time extra upfront payment of \$8,000.
2. A three-year lease at \$15,000/y.

Such lease options are commonly available, e.g., for cars see the [Bankrate.com Lease Calculator](#).

# Which Lease is Better?

- ▶ A costs \$32 total for 2 years
- ▶ B costs \$45 but is for 3 years.
- ▶ But A takes more money up-front.
- ▶ But B is cheaper per year.

# How Can You Compare Leases?

- ▶ What is the “equivalent annual rent” of Lease A?

# Comparing Leases — Assumptions?

- ▶ What if you need to use the building for exactly 2 years (no sublets!)?
- ▶ What if you need to use the building for exactly 3 years?
- ▶ What if a 3-year old building is worse than a 2-year old building?

# APPENDIX

Chapter 2 also contains

- ▶ Proof of Formulas.
- ▶ The formula for a *growing annuity*, rarely needed.
  - ▶ Beloved only by sadistic finance professors for exam questions.